

VIRAL PROCESSES BY RANDOM WALKS ON RANDOM REGULAR GRAPHS

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We study the SIR epidemic model with infections carried by k particles making independent random walks on a random regular graph. Here we assume $k \leq n^\epsilon$, where n is the number of vertices in the random graph and ϵ is some sufficiently small constant. We give an edge-weighted graph reduction of the dynamics of the process that allows us to apply standard results of Erdős-Rényi random graphs on the particle set. In particular, we show how the parameters of the model give two thresholds: In the subcritical regime, $O(\ln k)$ particles are infected. In the supercritical regime, for a constant $\beta \in (0, 1)$ determined by the parameters of the model, βk get infected with probability β , and $O(\ln k)$ get infected with probability $(1 - \beta)$. Finally, there is a regime in which all k particles are infected. Furthermore, the edge weights give information about when a particle becomes infected. We exploit this to give a completion time of the process for the SI case.

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1. Introduction. The spread of an infection throughout a population, often referred to loosely as an *epidemic*, has come to be modeled in various ways in the literature, spurred by the richness of domains in which the notion of a virus has gone beyond the traditional biological phenomenon. Electronic viruses over computer networks are not the only extension, others include *rumour spreading* [25] or *broadcasting* [5] and *viral marketing* [4]. Models may vary over domains, but the underlying principle is one of spread of some unit of information or state through interaction between individuals.

In much of the literature on the spread of epidemics as well as the dissemination of information, individuals reside at fixed vertices of a graph and the evolution of the state of an individual depends on the state of its neighbours in the graph. In particular if the graph is complete, mean-field (homogeneous mixing) models have been exploited to study the outcome of diffusion process [11]. More recently, there has been an increasing interest in understanding the impact of the network topology on the spread of epidemics in networks with fixed nodes, see [15] for a review of such results. There has, however, been little analytical work to date on models where the possible interactions between the nodes are dynamic, i.e., the underlying network structure evolves in time.

We explore a particular instance of dynamic interaction by assuming that individuals are mobile particles and can only infect each other if they are in sufficiently close proximity. The model is motivated both by certain kinds of biological epidemics, whose transmission may be dominated by sites at which individuals gather in close proximity (e.g., workplaces or public transport for a disease like SARS, cattle markets for foot-and-mouth disease, etc.) and by malware. Furthermore, it is relevant to studying the dissemination

of information in opportunistic networks [7] where the information is transmitted between users who happen to be in each other's range. As in the case of static networks [25] one may be interested in the time it takes for the rumour to be known to all users.

In our model (elaborated upon below) there are k particles making independent, discrete-time, synchronous random walks on an n -vertex r -regular random graph G . Each particle is in one of three states: susceptible (S), infected (I), recovered (R). An infected particle can infect a susceptible particle, which remains infected for a fixed *infectious period* ξ before recovering permanently. This is known as the *SIR epidemic model* and is extensively studied in the literature. When $\xi = \infty$ (the *SI model*) particles never go from I to R.

Two questions can be asked: (1) When $\xi < \infty$, how many particles ever get infected? (2) When $\xi = \infty$, what is the *completion time* of the process, that is, how long till the last infection? We address both of these questions by reducing the dynamics of the process to what we term an *interaction graph*. This turns out to have the structure of an Erdős-Rényi (E-R) random graph $\mathcal{G}_{k,\hat{q}}$ on the set of particles, where the edge probability $\hat{q}$ is a function of the parameters of the model. Infected particles are connected components in $\mathcal{G}_{k,\hat{q}}$, and so well known results from the literature on E-R random graphs can be applied using our reduction to answer question (1). In particular, we show how the parameters of the model produce two thresholds: In the subcritical regime, $O(\ln k)$ particles are infected with high probability (**whp**), that is, with probability tending to 1 as $n \rightarrow \infty$. In the supercritical regime, for a constant β determined by the parameters of the model, βk get infected with probability β , and $O(\ln k)$ get infected with probability $(1 - \beta)$. Finally, there is a regime in which all k particles are infected **whp**.

Furthermore, the interaction graph reduction assigns weights on the edges that give information about when a particle becomes infected. This information can be used for addressing question (2), which we do by giving a convergence in probability. This is detailed in Theorem 2.

Whilst the metaphor of an epidemic is a motivating and illustrative one, this work is part of the more general scope of *interacting particle systems* (see, e.g., [2, ch. 14]).

2. Model. Let $r \geq 3$ be fixed. Let $\mathcal{G}_r$ denote the set of r -regular graphs with vertex set $V_G = \{1, 2, \dots, n\}$ and the uniform measure. Let $G = (V_G, E_G)$ be chosen uniformly at random (**uar**) from $\mathcal{G}_r$. The results in this paper are always asymptotic in $n = |V_G|$. The notation O, o, Ω, Θ have their usual meanings with respect to n . We denote by $\Omega^+(1)$ a quantity that

can be replaced by an arbitrarily large constant. In some contexts where the sign does not matter, we may write $+o(1)$ instead of $-o(1)$, so as not to lend any significance to the sign. It will be obvious when this is the case.

Denote the set of particles in the system by $\mathcal{P}$. At step t , let $\mathcal{S}(t), \mathcal{I}(t), \mathcal{R}(t)$ be the set of susceptible, infected and recovered particles respectively. Since a given particle is in precisely one of these sets at a given time, they form a partition of $\mathcal{P}$.

When two particles x and y meet at some vertex $v \in G$, we call that an *xy meeting*. When there is an *xy meeting* at some time step t , an *interaction* takes place between them at that time step with probability $\rho \in (0, 1]$, which is a fixed constant parameter of the model. We term such an event an *xy interaction* and call ρ the *interaction probability*. If one of the particles is infected and the other is susceptible, the infection takes place upon interaction. The infectious period ξ is not restricted to being finite or constant; it is permitted to be ∞ or some function of n , for example.

Consider that the time step counter has just changed from $t-1$ to t : every particle x makes an independent move in its random walk. Subsequently, the rules are as follows:

1. If x, y are on the same vertex v , there is an *xy interaction* with probability ρ ; if they are on different vertices they cannot interact; each particle pair (non)interaction is independent of every other particle pair (non)interaction.
2. If $x \in \mathcal{S}(t-1)$ then $x \in \mathcal{S}(t)$, unless there was an *xy interaction* with at least one particle $y \in \mathcal{I}(t-1)$. In the latter case, we say x was infected at time $t-1$ and write $t(x) = t-1$.
3. If x was infected at time $t(x)$, then $x \in \mathcal{I}(t)$ for $t = t(x) + 1, t(x) + 2, \dots, t(x) + \xi$. Subsequently, $x \in \mathcal{R}(t)$ for all $t > t(x) + \xi$.

(note, we assume that $\mathcal{R}(0) = \emptyset$, and so a particle could only be in $\mathcal{R}(t)$ if it had been infected at some time previous to t).

Observe two things from the above rules. Firstly, infections are not transitive in a single time step. For example, suppose $x \in \mathcal{I}(t-1)$ meets $y, z \in \mathcal{S}(t-1)$ at vertex v at time step t . If x interacts with y but not z , then y does not pass on the infection to z at that time step, regardless of whether or not they interact.

Secondly, the rules allow, in principle, for infections to be passed on even with infectious period $\xi = 1$; there could be a chain of infections with x infecting y , which in turn infects z , etc.

A note on notation and terminology: When we refer to a ‘period’ $[t_1, t_2]$, we mean the set of time steps $\{t_1, t_1 + 1, t_1 + 2, \dots, t_2\}$. We may have infinite

periods, e.g., $[T, T + 1, \dots]$. When we refer to “time t ” we are referring to step t on the counter – this is a discrete-time process. The first time step is $t = 1$, and $t = 0$ is not a time step, but a useful convention to express the initial condition.

3. Assumptions.

We first specify some assumptions of the model. If each particle is at distance at least $\omega(k, n) \equiv \Omega(\ln \ln n + \ln k)$ from every other then we say the particles are in *general position* (g.p.).

We assume the following: **(i)** G is typical (“typical” is defined in section 7.1). **(ii)** The number of particles $k \leq n^\epsilon$ where $\epsilon > 0$ is some sufficiently small constant. **(iii)** Particles start in general position. **(iv)** $\mathcal{I}(0) = \{x_0\}$, and we refer to x_0 as the *initial infective*. Of course we also assume $|\mathcal{R}(0)| = 0$.

A graph G is *typical* if it satisfies certain conditions. The definition will be given in following sections but for now it suffices to say that most graphs in $\mathcal{G}_r$ are typical, that is, a graph G picked **uar** from $\mathcal{G}_r$ will be typical **whp**.

Assumption **(iii)** is also not unreasonable; it is straightforward to verify that if the positions of each of the k particles are chosen **uar** from the vertices of G , then **whp** they will be in g.p. with respect to each other if ϵ is small enough.

It is not difficult to extend our results to a greater number of initially infected particles, but we make assumption **(iv)** for convenience and clarity.

4. Results.

Let M_k be the total number of particles that ever get infected in the course of the process, and let T_k be the *completion time* of the process, the time step at which the last infection takes place. Define

$$(1) \quad \theta_r = \frac{r-1}{r-2}$$

and

$$(2) \quad \psi = \frac{\rho(r-1)}{r-2+\rho}$$

(observe that $\rho \leq \psi \leq 1$).

THEOREM 1. *Assume the conditions of section 3 and suppose $k \rightarrow \infty$ as $n \rightarrow \infty$. Let*

$$(3) \quad \phi = k \left(1 - \left(1 - \frac{\psi}{\theta_r n} \right)^\xi \right).$$

(i) *If $\phi < 1$ then **whp**, $M_k = O(\ln k)$.*

- (ii) If $\phi \rightarrow c$ for any constant $c > 1$, then with probability $(1 + o(1))\beta$, $M_k = (1 + o(1))\beta k$, otherwise $M_k = O(\ln k)$. β is the unique solution in $(0, 1)$ of the equation $1 - x = e^{-cx}$.
- (iii) If $\phi > (1 + \varepsilon) \ln k$ where $\varepsilon > 0$ is a constant, then **whp** $M_k = k$ (i.e., all the particles get infected).

Theorem 1 is for finite ξ , but observe that taking the convention that $x^\infty = 0$ for $|x| < 1$ means that part (iii) is consistent with the SI model, for which all particles get infected almost surely. The theorem effectively gives conditions for transitions between different possible “regimes” of behaviour. The most interesting is the regime of part (ii), which is entered as ϕ transitions from $\phi < 1$ to $\phi > 1$. Roughly speaking, in this transition the number of infected particles goes from very few ($O(\ln k)$) **whp**, to a constant fraction βk with probability β , or $O(\ln k)$ with probability $1 - \beta$.

Concerning the completion times, we shall demonstrate that for $\xi = \infty$, (that is, the SI model) how the edge weightings can be exploited by application of a theorem of [19] to get the following.

THEOREM 2. *Assume the conditions of section 3 and suppose $k \rightarrow \infty$ as $n \rightarrow \infty$. When $\xi = \infty$, we have the following convergence in probability:*

$$(4) \quad \frac{T_k}{n \ln k/k} \xrightarrow{p} 2 \frac{\theta_r}{\psi},$$

where T_k is the completion time for k particles, that is, the time at which the final particle is infected.

5. Related Work. In this section, we briefly describe some of the relevant related work on diffusion processes like epidemic spreading and the dissemination of information in mobile environments.

There has recently been a growing body of work in the interacting particle systems community analysing epidemic models with mobile particles. In [12] the authors provide a review of the results, techniques and conjectures when the graph is an infinite lattice. In [26], the authors explore by means of mean-field approximations the evolution of the number of infected individuals when individuals perform random motions on the plane. Recent papers by Peres et al [24], Pettarin et al [27] and Lam et al [22] analyse mobile networks modeled as multiple random walks; as Brownian motion on $\mathbb{R}^d$ in [24], as walks on a 2-dimensional grid in [27] and as walks on a grids of dimension 3 and above in [22]. In each case, there is a parameter r within which distance a pair of walks can communicate, producing a communication graph (which is disconnected below a certain threshold r_c). [24] studies various aspects of

the communication graph, such as how long it takes a particular point to become part of the giant component. [27] studies the broadcast time T_B of a piece of information and originating in one agent in relation to r . Setting $r = 0$ means information is passed only when particles are coincident. In this case, T_B is equivalent to our completion time and [27] give, for $k \geq 2$, $T_B = \tilde{\Theta}(n/\sqrt{k})$ **whp**¹. In [22] $r = 1$ and the results they have show a significant difference to the 2-dimensional case of [27]. We state their results for the 3-dimensional case: There exists a constant c such that if $cn^{\frac{1}{3}} \log^2 n^{\frac{1}{3}} < k < n$ then $T_B = \tilde{\Theta}(n^{\frac{5}{6}}/\sqrt{k})$; if $k < cn^{\frac{1}{3}} \log^{-2} n^{\frac{1}{3}}$ then $T_B = \tilde{\Theta}(n/k)$. Statements are **whp**.

Of closer relevance to this work are [13] and [14]. Both of these papers study infections carried by random walks on graphs. In particular [13] analyses an SI model similar to ours; multiple random walks on a graph G that carry a virus, and the quantity of interest is the completion time. They give a general upper bound $\mathbf{E}[T_k] = O(m^* \ln k)$ for any graph G , where m^* is the expected meeting time of a pair of random walks maximised over all pairs of starting vertices. Special cases are analysed too, in particular, they give an upper bound of $\mathbf{E}[T_k] = O(\frac{nr}{k} \ln k \ln n)$ for random r -regular graphs. This is a factor $\ln n$ larger than the precise value of the process considered in this paper.

Finally, we mention [5], which studies flooding on dynamic random networks. A fixed set of n vertices is part of a dynamic random graph process where each edge is an independent two-state Markov chain, either existing or not existing. A single initial vertex initially holds a message and any vertex which receives this message broadcasts it on existing edges for the next k steps. Although flooding is a different process to multiple random walks, the authors develop a reduction to a weighted random graph with some similarity to the interaction graphs we present. It allows them to derive relations between the edge-Markov probabilities and state asymptotic bounds on the number of vertices that receive the message, as well as the broadcast (equivalently, completion) time.

6. Overview. Section 7 addresses the behavior of (multiple) random walks on a graph, drawing on established results from the literature. It introduces the product graph framework, used for mapping multiple walks on G to a single walk on the product graph H . Thus, analysis of the multiple walks can be done by analysis of the single walk on H . After the background theory is given, the *first visit lemma* is presented. This lemma, first estab-

¹The tilde notation hides polylogarithmic factors. e.g., $\tilde{O}(f(n)) = O(f(n) \log^c n)$ for some constant c .

lished in [8], then subsequently refined in other papers, gives the probability of a walk visiting a vertex v for the first time (after mixing) at step t . It is used to establish new lemmas, created specifically for the analysis of the problem in this paper. The culmination is Lemma 9, which calculates probabilities of meetings of particle pairs. This lemma is the main tool of section 7 that is used in subsequent sections.

In section 8 we introduce the interaction graph $\mathbf{I}$, a complete graph on the particle set $\mathcal{P}$ with edge weights determined by the outcome of the process. $\mathbf{I}$ can therefore be represented by a random $\binom{k}{2}$ -vector, some entries of which will be ∞ if their associated particles were never infected. The edge weights give timing information on particle infection times, encoding which particles get infected and when those infections occurred.

Our approach is to analyse special cases and build upon them until reaching the full general case. We start in section 9 with the SI model with $\rho = 1$, then generalise that in section 10 to the SI(R), keeping $\rho = 1$. Finally, in section 11, we give the most general case, SI(R) with $\rho \leq 1$. We reiterate that whilst ξ is permitted to be ∞ or take any positive integer value (which may vary with n), ρ is a constant.

A simple algorithm can be used to construct $\mathbf{I}$ from the unfolding of the process, but to actually calculate the probability of a particular realisation of $\mathbf{I}$, we will employ the tools of section 7, in particular, Lemma 9. This lemma gives the probability of a first meeting time of a pair of particles being at time t , but t is required to be at least $\ell = \Omega(T^3)$, where T is the mixing time of the walks. It cannot account for what happens in the first ℓ steps. As such, rather than calculate the weights of $\mathbf{I}$, we will couple the process to a slightly modified version of it that demands that the interaction probability ρ is temporarily switched to zero in the “blind” periods - those length- ℓ periods which cannot be accounted for - before being switched back. When $\mathbf{I}$ is constructed under the new process, it may turn out differently to what it would have been under the original process. Therefore, for the $\rho = 1$ case, we will use $\mathbf{I}'$ to specify the interaction graph under the modified process.

In section 9, where $\xi = \infty$, it will be shown that the edge weights of $\mathbf{I}'$ (which will all be finite, almost surely) are “almost” independent, being well approximated by independent and identically distributed (i.i.d.) random variables having geometric distribution with parameter $1/(\theta_r n)$, denoted $\text{Geom}(1/(\theta_r n))$. Thus, $\mathbf{I}'$ can be modeled by Λ , a complete graph with i.i.d random edge weights having that distribution. We can use Λ to calculate distances and relate these to $\mathbf{I}'$. We then need to show $\mathbf{I}'$ is a good approximation for $\mathbf{I}$. We do so by showing that **whp**, the two graphs, con-

structed under their respective processes, will give the same edge weights when the processes are coupled.

Section 9 concludes with the proof of Theorem 2 for the special case $\rho = \psi = 1$. This is done in section 9.3 by an application of a result of Janson [19] on distances in randomly edge-weighted graphs. The theorem is applied to Λ and transfers by the above arguments to $\mathbf{I}$.

In section 10, where we address $\xi < \infty$ (but keep $\rho = 1$), we try to quantify the size of the outbreak. A particle x is infected iff there is a ξ -path in $\mathbf{I}$ from the initial infective x_0 to x . A ξ -path is one in which all edge weights in the path are at most ξ . Equivalently, the infected particles form the connected component x_0 belongs to when we delete all edges in $\mathbf{I}$ with weights exceeding ξ . Referring to this connected component as $\mathcal{C}_{f_\xi(\mathbf{I})}$, we relate $\mathcal{C}_{f_\xi(\mathbf{I})}$ to $\mathcal{C}_{f_\xi(\Lambda)}$, the equivalent in Λ . The function $f_\xi(F)$ takes a graph F and returns the same graph, but with edges weighing more than ξ deleted. Ignoring edge weights, $f_\xi(\Lambda)$ is an Erdős-Rényi random graph on k vertices $\mathcal{G}_{k,\hat{q}}$ with edge probability $\hat{q} = 1 - (1 - \frac{1}{\theta_{rn}})^\xi$. Consequently, standard results on Erdős-Rényi random graphs give characterisations of the size of $\mathcal{C}_{f_\xi(\Lambda)}$, which in turn transfer to $\mathcal{C}_{f_\xi(\mathbf{I})}$ and subsequently to $\mathcal{C}_{f_\xi(\mathbf{I})}$. Thus, we can determine how many particles get infected as a function of the parameters of the model. This will give us Theorem 1 for the special case $\rho = 1$.

We then move on to section 11, where we deal with $\rho < 1$. We begin in section 11.1 with a heuristic treatment of a two particle system, consisting of the initial infective x_0 and another particle, which is susceptible. This will allow us to outline the techniques in a clear and concise way without being hindered by detail. Subsequently, in section 11.2, we will formalise the arguments given in section 11.1, extended to all k particles. The core of this section will be Lemma 23. This lemma is essentially a generalisation of Lemma 9. It will allow us to determine probabilities of having a first interaction at some time t , whilst allowing for the possibility that there may have been meetings of particles prior to t where no interaction took place. Section 11.2 builds on and generalises the previous ones where $\rho = 1$, and we will detail how Theorem 1 and Theorem 2 are justified in their full generality by the results in this section.

Finally, in section 12, we make concluding remarks, including possible extensions.

7. Random Walks on Graphs: Tools and Techniques. In this section we detail key concepts and lemmas that we use to analyse the viral process. In section 7.1 we give a formal definition of typical graphs. Knowing the properties of typical graphs, we can make statements about how walks

behave on them. In section 7.2 we describe how the long-term behaviour of a random walk - specifically its convergence to the stationary distribution - relates to the eigenvalues of its transition matrix. We also show how to map the k multiple walks on G on to a single walk on another graph, the *product graph* H . Much of the analysis of meeting times between walks is done through the framework of the product graph. A particular pair of particles meeting in G maps to the single walk on H being at a set of vertices of H . Such a set of vertices is contracted to a single vertex, resulting in a derived graph Γ upon which a single walk moves. Thus, calculations of particle meeting times in G are done by calculating the hitting times of a single walk to a contracted vertex in Γ .

Lemma 4 shows that for each of the graphs G, H, Γ , the walks are *rapidly mixing*, meaning that they converge to their respective stationary distributions quickly. This is a crucial component of the proofs and the behaviour of the processes. In section 7.3 we introduce a key lemma, Lemma 6, which allows us to make precise calculations of $f_t(u \rightarrow v)$, the probability that a walk starting at u visits v for the first time (after mixing) at time t . When this lemma is applied to contracted vertices in Γ , it gives us probabilities of meeting times of particles walking on G . Lemma 5 in section 7.3 gives formal justification for the use of vertex contraction to reason about visits to sets and section 7.4 investigates the probability that a particular vertex v was visited when a set of vertices S with $v \in S$ has been visited.

Lemma 6, in conjunction with Lemmas 5, 7 and 8 culminate in Lemma 9 in section 7.5. This lemma gives probabilities for a particular pair of particles meeting at time t and no other pair meeting before hand. It is the main tool used in section 9.

7.1. *Typical graphs.* Let

$$(5) \quad L_1 = \lfloor \epsilon_1 \log_r n \rfloor,$$

where $\epsilon_1 > 0$ is a sufficiently small constant.

A vertex v is *treelike* if there is no cycle in the subgraph $G[v, L_1]$ induced by the set of vertices within (graph) distance L_1 of v .

A cycle C is *small* if $|C| \leq L_1$.

- P1. G is connected and not bipartite.
- P2. The second eigenvalue of the adjacency matrix of G is at most $2\sqrt{r-1} + \epsilon$, where $\epsilon > 0$ is an arbitrarily small constant.
- P3. There are at most $n^{2\epsilon_1}$ vertices on small cycles.
- P4. No pair of cycles C_1, C_2 with $|C_1|, |C_2| \leq 100L_1$ are within distance $100L_1$ of each other.

We say an r -regular graph G is *typical* if it satisfies the properties P1–P4.

Note that P3 implies that at most n^{ϵ_C} vertices of a typical r -regular graph are not treelike, where

$$(6) \quad n^{\epsilon_C} = O(r^{L_1} n^{2\epsilon_1}) = O(n^{3\epsilon_1}).$$

LEMMA 3 ([10]). *Let $\mathcal{G}'_r \subseteq \mathcal{G}_r$ be the set of typical r -regular graphs. Then $|\mathcal{G}'_r|/|\mathcal{G}_r| \rightarrow 1$ as $n \rightarrow \infty$.*

P1 implies that a random walk will converge to a stationary distribution π on the vertex set. Because the graph is regular, π will be the uniform distribution. P2 implies that a random walk will converge quickly to the stationary distribution; it will be *rapidly mixing*. In fact, for all the graphs we consider, $O(k \ln n)$ steps is sufficient for our results. A typical graph also has most of its vertices treelike.

7.2. *Convergence to stationarity and product graph formulation.* Let G be a connected graph with n vertices and m edges. For random walk $\mathcal{W}_u$ starting at a vertex u of G , let $\mathcal{W}_u(t)$ be the vertex reached at step t . Let $P = P(G)$ be the matrix of transition probabilities of the walk, and let $P_u^t(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. If the random walk $\mathcal{W}_u$ on G is ergodic, it will converge to stationary distribution π . Here $\pi(v) = d(v)/(2m)$, where $d(v)$ is the degree of vertex v . We often write $\pi(v)$ as π_v . The eigenvalues of $P(G)$ are $\lambda_0 = 1 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq -1$. Let $\lambda_{\max} = \max(\lambda_1, |\lambda_{n-1}|)$. The rate of convergence of the walk is given by

$$(7) \quad |P_u^t(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_{\max}^t.$$

For a proof of this, see, for example, Lovász [23].

To ensure that the walk is both ergodic and that $\lambda_{\max} = \lambda_1$ we make the chain *lazy*; i.e., the walk only moves to a neighbour with probability $1/2$. Otherwise it stays where it is. This shifts each eigenvalue up by $1/2$, and so $\lambda_1 \geq \lambda_{n-1} = 0$, and (7) still holds.

Now define the *product graph* $H = H(G, k) = (V_H, E_H)$ to have vertex set $V_H = V^k$ and edge set $E_H = E^k$. The vertices $\mathbf{v}$ of H consist of k -tuples $\mathbf{v} = (v_1, v_2, \dots, v_k)$ of vertices $v_i \in V_G, i = 1, \dots, k$, with repeats allowed. Two vertices $\mathbf{v}, \mathbf{w}$ are adjacent if $(v_1, w_1), \dots, (v_k, w_k)$ are edges of G . The purpose of defining the graph H is that we can replace the k random walks $\mathcal{W}_{u_i}(t)$ on G with current positions v_i and starting positions u_i by a single walk $\mathcal{W}_{\mathbf{u}}^H(t)$. Note that because G is assumed to be simple, no vertex in V_G has a loop. Consequently, no edge $e = ((v_1, w_1), \dots, (v_k, w_k))$ in E_H has

$v_i = w_i$ for any $1 \leq i \leq k$. This is the case despite the actual walk on the graph being “lazy” for part of the time, as will be described below.

We introduce some extra notation: for a graph $F = (V_F, E_F)$, a vertex $v \in V_F$, and a set of vertices $S \subseteq V_F$, let $d_F(v)$ be the degree of vertex v in F , and let $d_F(S) = \sum_{v \in S} d_F(v)$.

Now, if $S \subseteq V_H$, then $\Gamma = \Gamma(S)$ is obtained from H by contracting S to a single vertex $\gamma(S)$. All edges, including loops and parallel edges are retained, producing a multigraph. Thus $d_\Gamma(\gamma) = d_H(S) = r^k |S|$. Moreover Γ and H have the same total degree $(nr)^k$, and the degree of any vertex of Γ , except γ , is r^k .

DEFINITION 1 (Mixing time, maximal mixing time). For $F \in \{G, H, \Gamma\}$, let $\mathcal{W}_u^F$ be a lazy random walk starting at $u \in V_F$. The mixing time T_F is the smallest t such that, for graph $F = (V_F, E_F)$ and $t \geq T_F$,

$$(8) \quad \max_{u, x \in V_F} |P_u^t(x) - \pi_x| \leq \frac{\min_{x \in V_F} (\pi_x)}{n^3}.$$

A maximal mixing time T is defined as

$$T = \max\{T_{\Gamma(S)} : S \subseteq V_H, d_H(S) \leq k^2 n^{k-1} r^k\}$$

Observe that the mixing time T_Γ depends on the particular set of vertices S that gets contracted. The maximal mixing time is defined for convenience; if there is ambiguity about what the mixing time is, or a single mixing time is stated for a number of contractions of sets S , then it is safe to assume the maximal mixing time.

The following is a slightly modified version of a lemma proved in [10].

LEMMA 4 ([10]). Let G be typical and let $S \subseteq V_H$ be such that $d_H(S) \leq k^2 n^{k-1} r^k$.

For $k \leq n$,

$$T_G = O(\ln n), \quad T_H = O(\ln n), \quad \text{and} \quad T_\Gamma = O(k \ln n).$$

As a consequence, a maximal mixing time T has $T = O(k \ln n)$.

We analyse our walks in the product graph and assume that we keep the chain lazy for the duration of the mixing time. At this point it is mixed, and we can stop being lazy. A lazy walk in the product graph maps to a process where all the walks move or don't move together. That is, with probability $1/2$, each walk independently takes a random step, and with probability $1/2$ none of them do. Consider the following two conditions: (i) interactions

can only take place upon moving to a new vertex, and (ii) ξ can only be decreased (by 1) upon moving to a new vertex. It is not difficult to see (e.g., through coupling) that the laziness of the walk cannot affect the infection outcomes. Laziness affects time, but only during mixing periods since we do not keep the chain lazy thereafter.

The following lemma formalises the notion that we can deal with a first visit (after the mixing time) to a member of a set S of vertices of a graph H by contracting S into a single vertex $\gamma = \gamma(S)$ and instead deal with a first visit to γ on this altered graph.

LEMMA 5 ([9]). *Let G be typical, $S \subseteq V_H$ be such that $d_H(S) \leq k^2 n^{k-1} r^k$ and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Let $\mathcal{W}_u^H$ be a random walk in H starting at $u \notin S$, and let $\mathcal{W}_u^\Gamma$ be a random walk in Γ starting at the same vertex $u \neq \gamma$. Let T be a mixing time satisfying (8) in both H and Γ . Let $\mathcal{A}_w(t)$ be the event that no visit was made to w in the period $[T, T+1, \dots, t]$. Then

$$\Pr(\mathcal{A}_\gamma(t); \Gamma) = \Pr(\bigwedge_{v \in S} \mathcal{A}_v(t); H) \left(1 + O\left(\frac{1}{n^3}\right) \right),$$

where the probabilities are those derived from the walk in the given graph.

7.3. *First visit lemma.* In this section, we introduce existing results about ‘first visit’ behaviours for random walks on graphs. For the graphs we consider, Lemma 6 below, the *first visit lemma* was proved in [10]. It was initially presented in [8] then refined and applied in a series of subsequent papers, amongst them [9, 10].

Below, for some $\epsilon > 0$, we denote by the term $\Omega^+(1)$ a quantity that is at least some positive constant $C(\epsilon)$, which can be made arbitrarily large by making ϵ sufficiently small.

LEMMA 6 (First visit lemma [8, 10]). *Let G be typical, $S \subseteq V_H$ be such that $d_H(S) \leq k^2 n^{k-1} r^k$ and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

For $F \in \{G, H, \Gamma\}$ let $T = T_F$ and $\mathcal{W}_u^F$ be a walk started at $u \in V_F$. Let $f_t = f_t(u \rightarrow v)$ be the probability that the first visit of $\mathcal{W}_u^F$ to $v \in V_F$ in the period $[T, T+1, \dots]$ occurs at step t , and let $\mathcal{A}_v(t)$ be the event that $\mathcal{W}_u$ does not visit v in the period $[T, T+1, \dots, t]$.

Let

$$(9) \quad \lambda = \frac{1}{KT}$$

for some sufficiently large constant K . Then, for all $t \geq T$,

$$(10) \quad f_t(u \rightarrow v) = (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^{t+1}} + O\left(T\pi_v e^{-\lambda t/2}\right),$$

and

$$(11) \quad \Pr(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + (1 + O(T\pi_v))\pi_v/R_v)^t} + O\left(T^2\pi_v e^{-\lambda t/2}\right),$$

where

$$(12) \quad p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}$$

and

$$(13) \quad R_v = \theta_r + O(k^2 n^{-\Omega(1)} + (k \ln n)^{-\Omega^+(1)}).$$

We briefly discuss the terms R_v in (12) and $\Omega^+(1)$. For a given graph, R_v is the expected number of returns in the mixing time to a vertex v , for a walk that starts at v . The initial placement of the walk at v at time $t = 0$ is counted. To usefully apply the lemma, one needs to calculate (or approximate) R_v . Consider the case where v is a vertex of a r -regular random graph G . The walk is rapidly mixing (the mixing time T being $O(\ln n)$), and for most vertices v , the local structure is a tree. Because of this, the quantity $\theta_r = \frac{r-1}{r-2}$, the expected number of returns (ever) to the root of an r -regular infinite tree, provides a close approximation to R_v on G . For $R_{\gamma(S)}$, the expected number of returns in the mixing time to a contracted vertex $\gamma(S)$ in the product graph, our bound on the mixing time T_Γ is $O(k \ln n)$. However, it turns out that θ_r also provides a good approximation for $R_{\gamma(S)}$. This was determined in [10] and our statement of the first visit lemma incorporates both the general statement introduced in [8], as well as the bound on $R_{\gamma(S)}$ given in [10]. The fact that the $\Omega^+(1)$ term in (13) can be an arbitrarily large constant is demonstrated in the derivation of R_v in [10, Lemma 19].

A consequence of the condition $d_H(S) \leq k^2 n^{k-1} r^k$ is that $d(\gamma(S)) \leq k^2 n^{k-1} r^k$, so $\pi_{\gamma(S)} \leq k^2/n$. Since $R_{\gamma(S)} = (1 + o(1))\theta_r$, we have $p_{\gamma(S)} = O(\pi_{\gamma(S)}/R_{\gamma(S)}) = O(\pi_{\gamma(S)}) = O(k^2/n)$. Therefore, $T\pi_{\gamma(S)} = O(k^3 \ln n/n) = o(1)$, if $k \leq n^\epsilon$ and ϵ is small enough.

We will re-write (10) and (11) in a form that is more natural in the context of this paper, in particular, a form that resembles that of a geometric distribution. First note that

$$((1 - p_v)(1 + p_v))^t = (1 - p_v^2)^t = 1 - O(p_v^2 t),$$

so

$$\begin{aligned} \frac{1}{(1+p_v)^t} &= \frac{(1-p_v)^t}{1-O(p_v^2 t)} \\ &= (1+O(p_v^2 t))(1-p_v)^t \\ &= (1+O(\pi_v^2 t))(1-p_v)^t. \end{aligned}$$

In the above we have used the fact that $p_v = \Theta(\pi_v)$ since $R_v = (1+o(1))\theta_r$ for the graphs in this paper.

We re-write (10) as follows:

$$\begin{aligned} f_t(u \rightarrow v) &= (1+O(T\pi_v)) \frac{p_v}{(1+p_v)^{t+1}} + O\left(T\pi_v e^{-\lambda t/2}\right) \\ &= (1+O(T\pi_v))(1-O(p_v)) (1+O(\pi_v^2 t)) p_v (1-p_v)^t + O\left(T\pi_v e^{-\lambda t/2}\right) \\ &= \left[1+O(T\pi_v) + O(\pi_v^2 t) + O\left(Te^{-\lambda t/2}(1-p_v)^{-t}\right)\right] p_v (1-p_v)^t. \end{aligned}$$

Now

$$\begin{aligned} e^{-\lambda t/2}(1-p_v)^{-t} &= \left(e^{\frac{1}{2KT}}(1-p_v)\right)^{-t} \\ &\leq \left(e^{\frac{1}{2KT}}e^{-2p_v}\right)^{-t}, \end{aligned}$$

and $\frac{1}{2KT} - 2p_v = \Omega(1/T)$ since, as discussed above, $T\pi_v = o(1)$.

Thus, we can write (10) in the form

$$(14) \quad f_t(u \rightarrow v) = \left[1 + O(T\pi_v) + O(k^4 t/n^2) + O\left(T^2 e^{-\Omega(t/T)}\right)\right] p_v (1-p_v)^t.$$

Similarly, (11) can be written as

$$(15) \quad \mathbf{Pr}(\mathcal{A}_v(t)) = \left[1 + O(T\pi_v) + O(k^4 t/n^2) + O\left(T^2 e^{-\Omega(t/T)}\right)\right] (1-p_v)^t.$$

7.4. *Which vertex in the set S was visited?* For a set of vertices S , the following lemma gives the probability that a particular vertex $v \in S$ is visited when S is visited for the first time after the mixing time.

LEMMA 7. *Let G be typical, $S \subseteq V_H$ be such that $d_H(S) \leq k^2 n^{k-1} r^k$ and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Let γ be the contraction of S in H , and for $v \in S$, let δ be the contraction of $S \setminus \{v\}$ in H , resulting in graphs $\Gamma(S)$ and $\Gamma(S \setminus \{v\})$, respectively. Let T be the mixing time satisfying (8) in both $\Gamma(S)$ and $\Gamma(S \setminus \{v\})$. Let p_v, p_γ

and p_δ be as given by (12) for v, γ, δ in their respective graphs. Let ϵ_v be the solution to $p_\gamma - p_\delta = p_v(1 + \epsilon_v)$.

For $t \geq 2(T + L)$ where $L = T^3$, let $\mathcal{B}_v = \mathcal{B}_v(t)$ be the event that the first visit to S in the period $[T, T + 1, \dots]$ occurs at step t and that the visit is to node $v \in S$. Then,

$$(16) \quad \Pr(\mathcal{B}_v) = \left(1 + (1 + o(1))\epsilon_v + O(L\pi_\gamma) + O\left(\frac{k^4 t}{n^2}\right) \right) p_v(1 - p_\gamma)^t.$$

Note that when v is connected only to other vertices in S , it must be that $\Pr(\mathcal{B}_v) = 0$. The RHS of (16) is consistent with this since in such a case, $\epsilon_v = -1 \pm o(1)$.

PROOF OF LEMMA 7. It is enough to prove the lemma for a two vertex set $S = \{u, v\}$, as one vertex can always be a contraction of a set. Let t be expressed as $t = 2T + L + s$, where $s \geq L$. Divide $[0, t]$ into successive intervals of length T, s, T, L respectively, i.e. $[0, T - 1]$, $[T, s + T - 1]$, $[s + T, s + 2T - 1]$, $[s + 2T, t]$.

Let $\mathcal{A}$ be the event that $\mathcal{W}(\sigma) \notin \{u, v\}$ for $\sigma \in [T, s + T - 1]$ and that $\mathcal{W}(t) = u$, but $\mathcal{W}(\sigma) \neq u$ for $\sigma \in [s + 2T, t - 1]$. Contract S to make $\gamma = \gamma(S)$ in $[T, T + s - 1]$. Applying (15) to the period $[T, s + T - 1]$, i.e., letting $t = s + T - 1$, and noting that $T = O(k \ln n)$ and $s \geq L = T^3$

$$\begin{aligned} \Pr(\mathcal{A}_\gamma(s + T - 1)) &= \left(1 + O(T\pi_\gamma) + O\left(\frac{k^4 s}{n^2}\right) + O\left(T^2 e^{-\Omega(T^2)}\right) \right) (1 - p_\gamma)^{s+T-1} \\ &= \left(1 + O(T\pi_\gamma) + O\left(\frac{k^4 s}{n^2}\right) \right) (1 - p_\gamma)^{s+T-1} \end{aligned}$$

Now, starting from some vertex x at time $s + T$, we apply (14) for u to not be visited in the period $[s + 2T, t - 1]$ then be visited at t :

$$f_{t-s-T}(x \rightarrow u) = \left(1 + O(T\pi_u) + O\left(\frac{k^4 t}{n^2}\right) + O\left(T^2 e^{-\Omega((t-s-T)/T)}\right) \right) (1 - p_u)^{t-s-T} p_u.$$

Noting $t - s - T = L + T \geq T^3$,

$$f_{t-s-T}(x \rightarrow u) = \left(1 + O(T\pi_u) + O\left(\frac{k^4 t}{n^2}\right) \right) (1 - p_u)^{L+T} p_u.$$

Multiplying them together,

$$\begin{aligned} \Pr(\mathcal{A}) &= \Pr(\mathcal{A}_\gamma(s + T - 1)) f_{t-s-T}(x \rightarrow u) \\ &= \left(1 + O(T\pi_\gamma) + O\left(\frac{k^4 t}{n^2}\right) \right) (1 - p_\gamma)^{s+T-1} (1 - p_u)^{L+T} p_u \\ &\leq \left(1 + O(T\pi_\gamma) + O\left(\frac{k^4 t}{n^2}\right) \right) (1 - p_\gamma)^s (1 - p_u)^L p_u. \end{aligned}$$

Now,

$$\begin{aligned}
(1 - p_\gamma)^s (1 - p_u)^L &= (1 - p_\gamma)^t \left(\frac{1 - p_u}{1 - p_\gamma} \right)^L (1 - p_\gamma)^{-2T} \\
&= (1 - p_\gamma)^t \left(1 + \frac{p_\gamma - p_u}{1 - p_\gamma} \right)^L (1 + O(T\pi_\gamma)) \\
&= (1 - p_\gamma)^t (1 + O(L\pi_\gamma)) (1 + O(T\pi_\gamma)) \\
&= (1 - p_\gamma)^t (1 + O(L\pi_\gamma)).
\end{aligned}$$

Let $\mathcal{B}_u$ be the event that $\mathcal{W}(t) = u$ and $\mathcal{W}(\sigma) \notin \{u, v\}$ for $\sigma \in [T, t-1]$. Then $\mathcal{B}_u \subseteq \mathcal{A}$ and so $\Pr(\mathcal{B}_u) \leq \Pr(\mathcal{A})$. It follows that

$$(17) \quad \Pr(\mathcal{B}_u) \leq p_u (1 - p_\gamma)^t \left(1 + O(L\pi_\gamma) + O\left(\frac{k^4 t}{n^2}\right) \right).$$

However, by contracting S we have that

$$\Pr(\mathcal{B}_u \cup \mathcal{B}_v) = (1 + O(T\pi_\gamma) + O(k^4 t/n^2)) p_\gamma (1 - p_\gamma)^t,$$

and so

$$\begin{aligned}
(18) \quad \Pr(\mathcal{B}_v) &\geq \Pr(\mathcal{B}_u \cup \mathcal{B}_v) - \Pr(\mathcal{B}_u) \\
&\geq (1 + O(L\pi_\gamma) + O(k^4 t/n^2)) (p_\gamma - p_u) (1 - p_\gamma)^t \\
&= (1 + (1 + o(1))\epsilon_v + O(L\pi_\gamma) + O(k^4 t/n^2)) p_v (1 - p_\gamma)^t.
\end{aligned}$$

The result follows from (17) and (18). $\square$

7.5. Particle pair meetings. Consider the (unordered) pair of particles (x, y) , $x, y = 1, \dots, k$. Particles x and y being at the same vertex in G maps in the product graph to a set of vertices $S = \{\mathbf{v} = (v_1, v_2, \dots, v_k) : v_x = v_y\} \subset V_H$. We can, therefore, calculate the probability of an xy meeting in G at time t by calculating the probability of the single random walk $\mathcal{W}_{\mathbf{u}}^H$ on H visiting S at time t . This in turn is done by the contraction described above, and calculating the probability of the walk $\mathcal{W}_{\mathbf{u}}^\Gamma$ on Γ visiting $\gamma(S)$ at time t . These two times are asymptotically equal by Lemma 5.

More generally, let $A \subseteq \{(x, y) : x, y \in \mathcal{P}, x \neq y\}$ be a set of particle pairs. Consider the event $\{\text{for some } (x, y) \in A \text{ there is an } xy \text{ meeting at time } t\}$. In the product graph this maps to the event $\{\mathcal{W}_{\mathbf{u}}^H(t) \in S\}$ where $S = \{\mathbf{v} = (v_1, v_2, \dots, v_k) : v_x = v_y \text{ for some } (x, y) \in A\}$.

To use Lemma 6 for the walk $\mathcal{W}_{\mathbf{u}}^\Gamma$ on Γ visiting $\gamma(S)$, we need to calculate the relevant p_v as per (12).

LEMMA 8. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

For a set of particle pairs A , let $S(A) \subset V_H$ be such that $\mathbf{v} = (v_1, v_2, \dots, v_k) \in S$ iff, for some pair $(x, y) \in A$, $v_x = v_y$ where v_x (resp. v_y) is the position of particle x (resp. y) in G . Then in $\Gamma = \Gamma(S(A))$,

$$(19) \quad p_\gamma = \frac{|A|}{\theta_r n} \left(1 - O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right)$$

where $\gamma = \gamma(S(A))$.

PROOF. Let $N = |S(A)|$. A particular pair $(x, y) \in A$ can be on n possible different vertices in G , and for each one, the other particles can be on n^{k-2} . Thus $N \leq |A|n^{k-1}$. Further, $N \geq N'$ where N' is the number of $\mathbf{v} \in H$ such that only one of the particle pairs occupy the same node of the graph, and

$$N' \geq |A|n^{k-1} - |A|^2 n^{k-2} = |A|n^{k-1} \left(1 - O \left(\frac{k^2}{n} \right) \right).$$

Thus, $N = |A|n^{k-1} \left(1 - O \left(\frac{k^2}{n} \right) \right)$ and since each vertex of H has degree r^k and contraction preserves degree,

$$\pi_\gamma = \frac{|A|r^k n^{k-1}}{n^k r^k} \left(1 - O \left(\frac{k^2}{n} \right) \right) = \frac{|A|}{n} \left(1 - O \left(\frac{k^2}{n} \right) \right),$$

and

$$T\pi_\gamma = O \left(\frac{|A|k \ln n}{n} \right) \left(1 - O \left(\frac{k^2}{n} \right) \right) = O \left(\frac{k^3 \ln n}{n} \right)$$

since $|A| \leq \binom{k}{2}$.

Hence (12) becomes

$$\begin{aligned} p_\gamma &= \frac{\frac{|A|}{n} \left(1 - O \left(\frac{k^2}{n} \right) \right)}{(\theta_r + O(k^2 n^{-\Omega(1)} + (k \ln n)^{-\Omega^+(1)})) \left(1 + O \left(\frac{k^3 \ln n}{n} \right) \right)} \\ &= \frac{|A|}{\theta_r n} \left(1 - O \left(\frac{k^3 \ln n}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right) \\ &= \frac{|A|}{\theta_r n} \left(1 - O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right) \end{aligned}$$

when ϵ is small enough. □

LEMMA 9. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Let $\ell = 2(T + T^3)$ where T is a maximal mixing time. For $t \geq \ell$, a set of particle pairs A and $(x, y) \in A$, let $\mathcal{B}_{(x,y)}(t)$ denote the following event: There is no ab meeting for any $(a, b) \in A$ in the period $[\ell, t - 1]$, and only xy meet at time t . Then

$$(20) \quad \mathbf{Pr}(\mathcal{B}_{(x,y)}(t)) = \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} + \frac{k^4 t}{n^2}\right)\right) p(1 - |A|p)^t,$$

where

$$(21) \quad p = \frac{1}{\theta_r n} \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}}\right)\right).$$

PROOF. Let $S \subset V_H$ be the set of vertices in H which correspond to x and y being incident in G , but no other pair $(a, b) \in A$ being incident. $|S| \leq n^{k-1}$ and $|S| \geq n^{k-1} - |A|^2 n^{k-2} = n^{k-1} \left(1 - O\left(\frac{k^4}{n}\right)\right)$. By similar calculations as in Lemma 8, we get

$$p_v = \frac{1}{\theta_r n} \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}}\right)\right)$$

where $v = \gamma(S)$.

We use Lemma 7 with Lemma 8. Referring to (16), $L\pi_\gamma = O(k^5(\ln n)^3/n)$ and

$$\epsilon_v = O\left(\frac{k^2}{n^{\Omega(1)}} + \frac{k^2}{(k \ln n)^{\Omega^+(1)}}\right) = O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}}\right).$$

Subsequently, (16) gives us (20). $\square$

Thus far, we have presented a set of lemmas that allow us to calculate probabilities for meetings between subset of the k particles. This was done via the product graph framework, that allowed us to map the state of the k walks on G to a single walk on H , then analyse meetings between walks on G in terms of the single walk on H visiting specific vertices. From here on, we can largely forget about H , and just use the lemmas we have established through it, in particular, Lemma 9, which will be the main tool used to calculate probabilities of outcomes of the process.

In the next section, we will describe the interaction graph framework, which allows us to map the unfolding of the process into a set of edge weights that capture timing and outbreak information. We will then use the tools from this section to calculate probabilities of particular interaction graphs being realised.

8. Interaction Graph. The *interaction graph* $\mathbf{I} = (\mathcal{P}, E_{\mathbf{I}})$ is a weighted complete graph on the particle set $\mathcal{P}$, thus $E_{\mathbf{I}} = \{(x, y) : x, y \in \mathcal{P}, x \neq y\}$. For a particle x , let $t(x)$ be the time at which x is infected, or ∞ if it never gets infected. For an *interaction edge* $e = (x, y) \in E_{\mathbf{I}}$, let $t(e) = \min\{t(x), t(y)\}$ (meaning $t(e) = \infty$ if neither x nor y gets infected). Then the weight $w_{\mathbf{I}}(e)$ of the edge is a random variable defined as

$$(22) \quad w_{\mathbf{I}}(e) = \begin{cases} \min\{t - t(e) : t > t(e), xy \text{ interaction at } t\} & \text{if } t(e) < \infty \\ \infty & \text{otherwise} \end{cases}$$

In the particular case that $\rho = 1$, an interaction happens with every meeting, in which case the edge weight represents the time elapsed between the infection and the next meeting.

For the SI model, i.e., $\xi = \infty$, every particle will get infected almost surely, and so no edge of $\mathbf{I}$ will have weight ∞ . This will not, in general, be the case for the SIR model.

We can think of a timer or clock associated with each interaction edge $e = (x, y)$. The clock becomes active when either x or y becomes infected, and stops when they next interact. Hence, when a particle x gets infected, one clock stops being active, and generally, some number become active simultaneously. The exception to the latter is when all other particles became infected before x , in which case, the clocks associated with x had already become active previously.

Thus, the weighted complete graph $\mathbf{I}$ can be represented by a random $\binom{k}{2}$ -dimensional vector $(w_{\mathbf{I}}(e_i))_{i=1, \dots, \binom{k}{2}}$ where the edges are labeled in some arbitrary order. A realisation of the graph $\mathbf{I}$ is a specific set of values $\mathbf{z} = (z_1, z_2, \dots, z_{\binom{k}{2}})$ for each random variable $w_{\mathbf{I}}(e_i)$.

9. The SI Model with $\rho = 1$. This section deals with the special case $\xi = \infty$, $\rho = 1$.

Two key ideas we use are that (1) the meeting times between pairs of particles are almost independent, and that (2) the meeting time for a pair of particles (x, y) is roughly distributed as $\text{Geom}(\frac{1}{\theta_r n})$, i.e., as the geometric distribution with parameter $\frac{1}{\theta_r n}$. We formalise (1) and (2) in Section 9.2.

Let F be a weighted graph on $\mathcal{P}$. For a particle $x \in \mathcal{P}$, denote by $d_F(x)$ the weighted distance (i.e., shortest weighted path length) from the initial infective x_0 to x . Furthermore, for an edge (i.e., particle pair) $e = (x, y)$, let $d_F(e) = \min\{d_F(x), d_F(y)\}$. The interaction graph allows us to relate weighted distance to infection time. Lemma 10 below holds for $\rho \leq 1$.

LEMMA 10. *When $\xi = \infty$ the following holds almost surely: For a particle x , $t(x) = d_{\mathbf{I}}(x)$, and for an edge $e \in E_{\mathbf{I}}$, $t(e) = d_{\mathbf{I}}(e)$.*

PROOF. For a particle x , let $P = (p_0 = x_0, p_1, p_2 \dots, p_l = x)$ be a shortest path from x_0 to x in $\mathbf{I}$. For $0 \leq i \leq l-1$, observe $t(p_{i+1}) \leq t(p_i) + w_{\mathbf{I}}(p_i, p_{i+1})$, thus, iterating, $t(x) \leq \sum_{i=0}^{l-1} w_{\mathbf{I}}(p_i, p_{i+1}) = d_{\mathbf{I}}(x)$.

Now, for x to get infected, there must be a chain of infections $Q = (x_0 = q_0, q_1, \dots, q_{l'} = x)$ from one particle to another starting from x_0 . For $0 \leq j \leq l'-1$, since q_{j+1} is infected by q_j , we have $t(q_{j+1}) = t(q_j) + w_{\mathbf{I}}(q_j, q_{j+1})$. Thus, iterating, we have $t(x) = \sum_{j=0}^{l'-1} w_{\mathbf{I}}(q_j, q_{j+1}) \geq d_{\mathbf{I}}(x)$. $\square$

In subsequent sections, this reduction is used to determine a completion time for the process and to derive another graph that gives the number of infected particles in the case $\xi < \infty$.

9.1. *(Almost) building an interaction graph.* One can build $\mathbf{I}$ simply by observing the process unfolding and starting and stopping clocks as it does so to determine edge weights. To *calculate* the probability of $\mathbf{I}$ taking a particular value (represented as a $\binom{k}{2}$ -vector of edge weights), we will use Lemma 9. However, this lemma only allows us to calculate the meeting times at values of $t \geq \ell = 2(T + T^3)$ where $T = O(k \ln n)$ is the mixing time. As such, we couple the process to a slightly modified version of it, and construct the interaction graph under the new process. We shall refer to this interaction graph as $\mathbf{I}'$.

Recall the role of clocks in determining edge weights of $\mathbf{I}$; the clock associated with $e = (x, y)$, $clock(e)$, is active precisely in the period $[t(e), t' - 1]$, where $t(e) = \min\{t(x), t(y)\}$ and t' is the first time after $t(e)$ that x and y interact. We describe the edge e as being *active* while $clock(e)$ is active. Thus there is a sequence of at most $\binom{k}{2}$ times $\tau_0, \tau_1, \tau_2, \dots, \tau_j$, in which the set of active edges changes. We call these times *epochs*. We let $\tau_0 = 0$, since at this point, x_0 has active edges with each of the other $k - 1$ particles.

A particle pair (a, b) is *incident* if a and b are at the same vertex in G . We parameterise the interaction probability so that $\rho(t)$ is the probability of interaction at time t , universally for all particle pairs. The new process makes the following modifications to the original one:

- (i) Set $\rho(t) = 0$ for $t \in [1, \ell]$
- (ii) If at time τ there was an interaction between an active pair (x, y) , set $\rho(t) = 0$ for $t \in [\tau + 1, \tau + \ell]$

These are the only differences - at all other times, we keep $\rho(t) = 1$. Furthermore, the definition of *active edge* remains the same for $\mathbf{I}'$; an edge

(particle pair) $e = (x, y)$ is active precisely in the period $[t(e), t' - 1]$, where $t(e) = \min\{t(x), t(y)\}$ and t' is the first time after $t(e)$ that x and y interact.

In the periods where $\rho(t) = 0$, the set of active edges does not change, and infections are not passed on, even if there were meetings between infected and susceptible particles. We call these *blind* periods, because setting $\rho(t) = 0$ is like ignoring interactions that may have otherwise occurred during those times. We can think of ℓ as an extended mixing time.

Now we define a weighted complete graph $\mathbf{I}' = (\mathcal{P}, E_{\mathbf{I}'})$: this is exactly $\mathbf{I}$ but with edge weights determined in the modified process. That is, for an edge $e = (x, y)$, the weight $w_{\mathbf{I}'}(e)$ of the edge is a random variable defined by the RHS of (22), but now $t(x)$ and $t(y)$ are the infection times assuming blind periods. Observe $w_{\mathbf{I}'}(e) > \ell$ because we still start the clock associated with it at time $t(e)$, but due to the blind periods, we will not stop it until the first time after $t(e) + \ell$ that they interact.

The modifications mean that in the sequence of epochs $\tau_0, \tau_1, \tau_2, \dots, \tau_j$ in which the set of active edges changes, we have $\tau_{i+1} - \tau_i > \ell$, and for each $i \in \{0, 1, \dots, j\}$ we set $\rho(t) = 0$ for $t \in [\tau_i + 1, \tau_{i+1} + \ell]$.

The initial stages of the construction of $\mathbf{I}'$ for the SI model with $\rho = 1$ is described as follows: We set $\rho(t) = 0$ for the first ℓ steps then set $\rho(t) = 1$, waiting until x_0 interacts with some particle. Suppose this happens with x_1 at time τ_1 . We label the edge $(x_0, x_1) \in \mathbf{I}'$ with τ_1 . We then $\rho(t) = 0$ at steps $\tau_1 + 1, \dots, (\tau_1 + \ell)$, then resume $\rho(t) = 1$, waiting for the next interaction to take place, either between x_0 and one of the remaining $k - 2$ particles, or x_1 and the remaining $k - 2$ particles. Suppose x_2 is the next amongst the remaining particles to interact, and this happens at time τ_2 . If it happens with x_0 , then the edge $(x_0, x_1) \in \mathbf{I}'$ has weight τ_2 . If it is with x_1 , then the edge $(x_1, x_2) \in \mathbf{I}'$ has weight $\tau_2 - \tau_1$. We then $\rho(t) = 0$ for the following ℓ steps then resume $\rho(t) = 1$ thereafter, waiting for the next relevant interaction.

Thus, the partial construction of $\mathbf{I}'$ at τ_2 in the former case has $\mathbf{I}' = \{w_{\mathbf{I}'}(x_0, x_1) = \tau_1, w_{\mathbf{I}'}(x_0, x_2) = \tau_2\}$, and in the latter case, it is $\mathbf{I}' = \{w_{\mathbf{I}'}(x_0, x_1) = \tau_1, w_{\mathbf{I}'}(x_1, x_2) = \tau_2 - \tau_1\}$. At time τ_2 , in the former case, x_0 has active clocks with $k - 3$ other particles, and each of x_1 and x_2 with $k - 2$ others (including each other). Suppose in this scenario, that the next interaction after $\tau_2 + \ell$ occurs between x_1 and x_2 , at τ_3 . Then $\mathbf{I}'$ at this point will be $\mathbf{I}' = \{w_{\mathbf{I}'}(x_0, x_1) = \tau_1, w_{\mathbf{I}'}(x_0, x_2) = \tau_2, w_{\mathbf{I}'}(x_1, x_2) = \tau_3 - \tau_1\}$.

Continuing in this manner, we will eventually build the complete edge-weighted graph $\mathbf{I}'$. *This will be exactly the same as $\mathbf{I}$ if, in the original process, there were no active pair interactions when $\rho(t) = 0$ in the modified process.* If, in fact, there were active pair interactions at those time steps

then $\mathbf{I}$ will be different to $\mathbf{I}'$ described above. However, we will show that, **whp**, this will not be the case. That is, **whp**, the graph $\mathbf{I}'$, we construct in this modified process, is the same as $\mathbf{I}$ that would have been constructed had we kept $\rho(t) = 1$ throughout.

9.2. Interaction graph approximation. We define a complete graph $\Lambda = (V_\Lambda, E_\Lambda)$ on the particle set $\mathcal{P}$ with i.i.d. random edge weights. The weight $w_\Lambda(e)$ of each edge $e \in E_\Lambda$ has distribution $\text{Geom}(q)$, where $q = \frac{\psi}{\theta_r n}$. In this section, $\psi = 1$ since $\rho = 1$.

Λ can be specified by a $\binom{k}{2}$ -vector; labelling the edges in Λ as e_i with $1 \leq i \leq \binom{k}{2}$, let z_i denote the realised weight of e_i and let $\Lambda = \mathbf{z}$ where $\mathbf{z} = (z_1, z_2, \dots, z_{\binom{k}{2}})$ denote this particular realisation of Λ . Then

$$(23) \quad \Pr(\Lambda = \mathbf{z}) = \Pr\left(\bigwedge_{i=1}^{\binom{k}{2}} w_\Lambda(e_i) = z_i\right) = \prod_{i=1}^{\binom{k}{2}} q(1-q)^{z_i-1}.$$

Our strategy will be as follows: We shall calculate the probability $\Pr(\mathbf{I}' = \mathbf{z})$ of a particular realisation $\mathbf{z}$ of $\mathbf{I}'$ by repeated application of Lemma 9. We shall show this probability is well-approximated by $\Pr(\Lambda = \mathbf{z})$, i.e., Λ serves as an “idealised” version of $\mathbf{I}'$ (and, in turn, $\mathbf{I}$). Hence, **whp** results in Λ can be transferred to $\mathbf{I}'$, and in turn transferred to $\mathbf{I}$ if the latter two are the same. Thus, to complete this proof strategy, we will show that $\mathbf{I}$ and $\mathbf{I}'$ are the same **whp**.

Assume the edges of $\mathbf{I}'$ and Λ are labeled in the same order. We shall show that for a class of $\mathbf{z}$ defined below as *good*, $\Pr(\Lambda = \mathbf{z})$ is a close approximation for $\Pr(\mathbf{I}' = \mathbf{z})$.

DEFINITION 2 (Good). *Let T be a maximal mixing time and let $\ell = 2(T + T^3)$. We say $\mathbf{z}$ is good when all of the components of $\mathbf{z}$ are finite and it satisfies the following:*

- (a) *If $\mathbf{I} = \mathbf{z}$ then none of the interactions that form the edges of $\mathbf{I}$ occur within ℓ steps of each other*
- (b) $\sum_{i=1}^{\binom{k}{2}} z_i \leq k^2 n \ln n$

We shall also refer to a graph F as good if $F = \mathbf{z}$ and $\mathbf{z}$ is good.

Part (a) of the above implies that when an infection occurs, no other infection takes place at that time step, nor during the following ℓ . We require this condition because we wish to apply Lemma 9, which only gives

probabilities for meetings that occur after ℓ steps. Therefore, we can only use Lemma 9 to calculate probabilities for interaction graphs in which all relevant interactions are separated by ℓ steps.

Part **(b)** is a technical condition. We require it because if weights z_i are too big, then our approximations will cease to hold.

As will be seen in Lemma 14, the probability of **I** having weights violating this condition goes to zero asymptotically.

We remind that in the context of **I'**, the notation $t(x)$ for a particle x refers to the time at which x is infected when we set $\rho(t) = 0$ in blind periods. Similarly, for an edge (particle pair) $e = (x, y)$, $t(e) = \min\{t(x), t(y)\}$. We may set $t(x) = \infty$ for a particle that never gets infected. In the present case where the infectious period $\xi = \infty$ (i.e., an SI model), every particle will get infected almost surely, but when we address $\xi < \infty$, this may not be the case, and the convention of setting “infection times” to ∞ will be convenient.

We shall use graph notation with vectors $\mathbf{z}$: For an edge e_i , $w_{\mathbf{z}}(e_i) = z_i$; for a particle y , $d_{\mathbf{z}}(y)$ is the weighted distance under $\mathbf{z}$ of y from x_0 ; for an edge $e = (x, y)$, $d_{\mathbf{z}}(e) = d_{\mathbf{z}}(x, y) = \min\{d_{\mathbf{z}}(x), d_{\mathbf{z}}(y)\}$.

Lemma 11 below holds for the SI model; when we consider the case $\xi < \infty$ we will give a generalisation of it.

LEMMA 11. *For good $\mathbf{z}$, consider the following:*

- (i) $\mathbf{I}' = \mathbf{z}$
- (ii) For each particle pair (edge) $e = (x, y)$, there is no xy interaction in the period, $[d_{\mathbf{z}}(e) + \ell, d_{\mathbf{z}}(e) + w_{\mathbf{z}}(e) - 1]$ and there is an xy interaction at time $\tau(e) = d_{\mathbf{z}}(e) + w_{\mathbf{z}}(e)$

Then (i) holds if and only if (ii) holds.

PROOF. $\mathbf{I}' = \mathbf{z}$ can be restated as: For each $e = (x, y)$, the first xy interaction after time $t(e) + \ell$ is at time $t(e) + w_{\mathbf{z}}(e)$. Hence, if we show that for each particle x , $t(x) = d_{\mathbf{z}}(x)$ then we are done.

(i) $\Rightarrow$ (ii) An equivalent to Lemma 10 holds for **I'**. The implication follows.

(ii) $\Rightarrow$ (i) Order particles by their distance from x_0 in $\mathbf{z}$: $x_0 = x_{(0)}, x_{(1)}, \dots, x_{(k-1)}$ where $i < j \Rightarrow d_{\mathbf{z}}(x_{(i)}) \leq d_{\mathbf{z}}(x_{(j)})$. We shall prove that $t(x_{(i)}) = d_{\mathbf{z}}(x_{(i)})$.

Clearly this proposition holds for $x_{(0)}$. Suppose for all $i \leq N - 1$, $t(x_{(i)}) = d_{\mathbf{z}}(x_{(i)})$. If $N = k$, we are done. Otherwise $N < k$. Let $x_{(M)}$ be a neighbour of $x_{(N)}$ on a shortest path from x_0 to $x_{(N)}$. Since $d_{\mathbf{z}}(x_{(M)}) < d_{\mathbf{z}}(x_{(N)})$, $M < N$, so by the induction hypothesis, $t(x_{(M)}) = d_{\mathbf{z}}(x_{(M)}) = d_{\mathbf{z}}(e)$ where $e = (x_{(M)}, x_{(N)})$. By (ii) this implies $t(x_{(N)}) \leq d_{\mathbf{z}}(e) + w_{\mathbf{z}}(e) = d_{\mathbf{z}}(x_{(N)})$.

Now consider the chain of infections starting at x_0 that led to $x_{(N)}$ being infected. Let $x_{(j)}$ be the first in the chain where $j > N-1$ and suppose it got infected by $x_{(i)}$, $i \leq N-1$. Then by **(ii)**, $t(x_{(j)}) = d_{\mathbf{z}}(x_{(i)}) + w_{\mathbf{z}}(x_{(i)}, x_{(j)}) \geq d_{\mathbf{z}}(x_{(j)}) \geq d_{\mathbf{z}}(x_{(N)})$, implying $t(x_{(N)}) \geq d_{\mathbf{z}}(x_{(N)})$.

Hence $t(x_{(N)}) = d_{\mathbf{z}}(x_{(N)})$ and the lemma follows. $\square$

Lemma 12 below holds for the SI model; when we consider the case $\xi < \infty$ we will give a generalisation of it.

LEMMA 12. *Assume the conditions of section 3.*

For good $\mathbf{z}$,

$$(24) \quad \Pr(\mathbf{I}' = \mathbf{z}) = (1 + o(1))\Pr(\Lambda = \mathbf{z}).$$

PROOF. By Lemma 11, $\mathbf{I}' = \mathbf{z}$ defines, for each edge $e = (x, y)$, a time $\tau(e) = d_{\mathbf{z}}(e) + w_{\mathbf{z}}(e)$. Letting $\tau_0 = t(x_0) = 0$, since $\mathbf{z}$ is good, we get a sequence of epochs $\tau_0 < \tau_1 < \tau_2 < \dots < \tau_{\binom{k}{2}-1}$ that are at least ℓ apart.

An edge e is active precisely in the period $[d_{\mathbf{z}}(e), d_{\mathbf{z}}(e) + w_{\mathbf{z}}(e) - 1]$ (and at no other time). Let A_i denote the set of active edges in the period $[\tau_i, \tau_{i+1} - 1]$. This defines a sequence $(A_i) = (A_0, A_1, \dots, A_{\binom{k}{2}-1})$ of active edge sets associated with epochs $\tau_0, \tau_1, \dots, \tau_{\binom{k}{2}-1}$ respectively, and which remain constant between epochs. The active set changes at each epoch, when an active edge is removed from the set, and possibly new ones are added. In general, an edge e will be a member of a number of active edge sets A_i .

Let $\sigma(\mathbf{z}) = ((e_{(1)}, \tau_1), (e_{(2)}, \tau_2), \dots, (e_{(\binom{k}{2})}, \tau_{\binom{k}{2}}))$, be defined by the above, where $e_{(i)}$ is the particle pair that interact at time τ_i . We shall use $\sigma(\mathbf{z})$ and (A_i) to calculate $\Pr(\mathbf{I}' = \mathbf{z})$. In particular, $\mathbf{I}' = \mathbf{z}$ if and only if both of the following hold.

- (1) if $(x, y) \in A_i$, there is no xy interaction in the period $[\tau_i + \ell, \tau_{i+1} - 1]$
- (2) particle pair $e_{(i)}$ interact at time τ_i , and this is the only pair in A_i that does.

The probability of the above will be determined by repeated application of Lemma 9 and the strong Markov property. Consider the process up until epoch τ_1 ; (1) dictates no interaction between any $(x_0, y) \in A_0 = \{(x_0, y) : y \in \mathcal{P}, y \neq x_0\}$ in the period $[\ell, \tau_1]$ and (2) dictates only an $e_{(1)} = (x_0, x_1)$ interaction at τ_1 . Applying Lemma 9:

$$(25) \quad \Pr(\mathcal{B}_{e_{(1)}}(\tau_1)) = \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} + \frac{k^4 \tau_1}{n^2}\right)\right) p(1 - |A_0|p)^{\tau_1}$$

Now if we consider the next period $[\tau_1, \tau_2 - 1]$, the set of active edges are $A_1 = \{(x_0, y) : y \in \mathcal{P}, y \neq x_0, x_1\} \cup \{(x_1, y) : y \in \mathcal{P}, y \neq x_0, x_1\}$. **(1)** dictates no interaction between any pair in A_1 and **(2)** dictates only an $e_{(2)}$ interaction at τ_2 . Thus we can apply Lemma 9 for this period and active edge set to get

$$\Pr(\mathcal{B}_{e_{(2)}}(\tau_2)) = \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} + \frac{k^4(\tau_2 - \tau_1)}{n^2}\right)\right) p(1 - |A_1|p)^{\tau_2 - \tau_1}.$$

Because $\mathbf{z}$ is good, $(\tau_{j+1} - \tau_j)$ is such that $k^4(\tau_{j+1} - \tau_j)/n^2 = n^{-\Omega(1)}$, which can be absorbed into the correcting factor of p , which has form (21). Therefore, we can write $\Pr(\mathcal{B}_{e_{(1)}}(\tau_1)) = p(1 - |A_0|p)^{\tau_1}$ and $\Pr(\mathcal{B}_{e_{(2)}}(\tau_2)) = p(1 - |A_1|p)^{\tau_2 - \tau_1}$.

We can continue in similar fashion for each epoch, and by the strong Markov property, we can multiply these probabilities to get

$$(26) \quad \Pr(\mathbf{I}' = \mathbf{z}) = \prod_{j=0}^{\binom{k}{2}-1} p(1 - |A_j|p)^{\tau_{j+1} - \tau_j},$$

where

$$p = \frac{1}{\theta_r n} \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}}\right)\right).$$

Letting $\Delta_j = \tau_{j+1} - \tau_j$, (26) can be written

$$\begin{aligned} \Pr(\mathbf{I}' = \mathbf{z}) &= p^{\binom{k}{2}} \prod_{j=0}^{\binom{k}{2}-1} \exp\{-(1 + O(|A_j|p))|A_j|p\Delta_j\} \\ &= p^{\binom{k}{2}} \exp\left\{-(1 + O(k^2 p))p \sum_{j=0}^{\binom{k}{2}-1} |A_j|\Delta_j\right\}. \end{aligned}$$

Furthermore,

$$\sum_{j=0}^{\binom{k}{2}-1} |A_j|\Delta_j = \sum_{j=0}^{\binom{k}{2}-1} \sum_e \mathbf{1}_{\{e \in A_j\}} \Delta_j = \sum_e \sum_{j=0}^{\binom{k}{2}-1} \mathbf{1}_{\{e \in A_j\}} \Delta_j$$

and

$$\sum_{j=0}^{\binom{k}{2}-1} \mathbf{1}_{\{e \in A_j\}} \Delta_j = \sum_{j=0}^{\binom{k}{2}-1} \mathbf{1}_{\{e \in A_j\}} (\tau_{j+1} - \tau_j) = z_e,$$

because this sums over all intervals $[\tau_j, \tau_{j+1} - 1]$ in which edge e is active, that sum being z_e . Hence, $\sum_{j=0}^{\binom{k}{2}-1} |A_j| \Delta_j = \sum_e z_e = \sum_{i=1}^{\binom{k}{2}} z_i$.

Thus,

$$\mathbf{Pr}(\mathbf{I}' = \mathbf{z}) = p^{\binom{k}{2}} \exp \left\{ -(1 + O(k^2 p)) p \sum_{i=1}^{\binom{k}{2}} z_i \right\}.$$

From (23), we have

$$\mathbf{Pr}(\Lambda = \mathbf{z}) = \prod_{i=1}^{\binom{k}{2}} q(1-q)^{z_i-1} = q^{\binom{k}{2}} (1-q)^{-\binom{k}{2}} \exp \left\{ -(1 + O(q)) q \sum_{i=1}^{\binom{k}{2}} z_i \right\}.$$

Therefore,

$$\frac{\mathbf{Pr}(\mathbf{I}' = \mathbf{z})}{\mathbf{Pr}(\Lambda = \mathbf{z})} = \left(\frac{p(1-q)}{q} \right)^{\binom{k}{2}} \exp \left\{ ((1 + O(q))q - (1 + O(k^2 p))p) \sum_{i=1}^{\binom{k}{2}} z_i \right\}.$$

$$\begin{aligned} \left(\frac{p(1-q)}{q} \right)^{\binom{k}{2}} &= \left(1 + O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right)^{\binom{k}{2}} \\ (27) \quad &= 1 + O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right), \end{aligned}$$

where we have used the fact that $k \leq n^\epsilon$ for ϵ small enough.

Since

$$(1 + O(k^2 p))p - (1 + O(q))q = O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \frac{1}{n},$$

then by part (b) of the definition of *good* $\mathbf{z}$,

$$(28) \quad \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \frac{1}{n} \sum_{i=1}^{\binom{k}{2}} z_i < \frac{k^2 \ln n}{n^{\Omega(1)}} + \frac{k^2 \ln n}{(k \ln n)^{\Omega^+(1)}}.$$

Since the $\Omega^+(1)$ term in (27) and (28) is an arbitrarily large constant, they are $1 + o(1)$ and $o(1)$ respectively, when $k \leq n^\epsilon$ for sufficiently small ϵ . Thus, we conclude $\mathbf{Pr}(\mathbf{I}' = \mathbf{z})/\mathbf{Pr}(\Lambda = \mathbf{z}) = 1 + o(1)$. $\square$

To prove Lemma 14 below, we require the following minor adaptation of Lemma 20 in [10]:

LEMMA 13 ([10]). *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Suppose that particles start with minimum separation at least $d = \alpha(\ln \ln n + \ln k)$ where α is a constant. Let $\tau = O(T^3)$ where $T = O(k \ln n)$. Then

$$(29) \quad \mathbf{Pr}(a \text{ given pair of particles } x, y \text{ meet during } \tau) = O(\tau^2 / (r - 1)^{d/6}).$$

It should be noted that the α in Lemma 13 can be made as large as required by making ϵ sufficiently small.

Lemma 14 below holds for general ξ and $\rho \leq 1$.

LEMMA 14. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ . With high probability,*

- (a) *None of the interactions that form the finitely-weighted edges of $\mathbf{I}$ occur within ℓ steps of each other*
- (b) *The sum of the finite edge weights of $\mathbf{I}$ is at most $k^2 n \ln n$*

PROOF. It will be convenient to prove part (b) of the definition first.

(b) Consider an active pair of particles (x, y) . The expected number of meetings they have before interacting is $1/\rho$. The expected time between meetings is $O(n)$. Since walks and interactions are independent, we can multiply these quantities together to give a bound on the expected time till an interaction. Since ρ is assumed to be constant, this is $O(n)$.

The result now follows by linearity of expectation and Markov's inequality.

(a) Let A be a set of active particle pairs. We let them mix for ℓ steps, and suppose the first interaction occurs at some (random) time $t \geq \ell$. Let $S \subset V_H$ correspond to a meeting of at least one pair in A . Let $S' \subset S$ correspond to more than one pair being within distance d of each other. We wish to calculate $\mathbf{Pr}(\mathcal{W}_{\mathbf{u}}^H(t) \in S' \mid \mathcal{W}_{\mathbf{u}}^H(t) \in S \wedge \mathcal{W}_{\mathbf{u}}^H(\tau) \notin S \text{ for } \tau \in [\ell, t - 1])$.

Let $\gamma = \gamma(S)$ and $\gamma' = \gamma(S')$. By Lemma 8, $\pi_\gamma = (1 + o(1))|A|/n$. Furthermore, since $|S'| \leq \binom{|A|}{2} r^{2d} n^{k-2}$, $p_{\gamma'} = O(|A|^2 r^{2d} / n^2)$.

Applying Lemma 7²,

$$\begin{aligned}
 (30) \quad & \mathbf{Pr}(\mathcal{W}_{\mathbf{u}}^H(t) \in S' \wedge \mathcal{W}_{\mathbf{u}}^H(\tau) \notin S \text{ for } \tau \in [\ell, t-1]) \\
 &= \left(1 + O(L\pi_\gamma) + O(\epsilon_{\gamma'}) + O\left(\frac{k^4 t}{n^2}\right)\right) p_{\gamma'}(1 - p_\gamma)^t \\
 &= (1 + o(1))p_{\gamma'}(1 - p_\gamma)^t.
 \end{aligned}$$

The correcting factor in the last line holds because $t \leq k^2 n \ln n$ by part (b), and it is straightforward to show $\epsilon_{\gamma'} = o(1)$.

$$(31) \quad \mathbf{Pr}(\mathcal{W}_{\mathbf{u}}^H(t) \in S \wedge \mathcal{W}_{\mathbf{u}}^H(\tau) \notin S \text{ for } \tau \in [\ell, t-1]) = (1 + o(1))p_\gamma(1 - p_\gamma)^t,$$

so dividing (30) by (31) gives

$$\begin{aligned}
 Pr(\mathcal{W}_{\mathbf{u}}^H(t) \in S' \mid \mathcal{W}_{\mathbf{u}}^H(t) \in S \wedge \mathcal{W}_{\mathbf{u}}^H(\tau) \notin S \text{ for } \tau \in [\ell, t-1]) &= O\left(\frac{|A|r^{2d}}{n}\right) \\
 &= O\left(\frac{1}{n^{\Omega(1)}}\right).
 \end{aligned}$$

There are at most $\binom{k}{2}$ interactions determining the edge weights of $\mathbf{I}$; taking the union bound over all of them, this is $o(1)$ if ϵ is small enough.

The RHS of (29) is $O(1/(k \ln n)^{\Omega^+(1)})$. Consequently, we can apply Lemma 13 across all (at most $\binom{k}{2}$) particles pairs across all (at most $\binom{k}{2}$) interactions. We thus conclude that **whp**, at most one pair interact at any time and there are no interactions in any of the following length ℓ blind periods. $\square$

COROLLARY 15. *With high probability,*

- (i) *For general ξ , $\mathbf{I} = \mathbf{I}'$*
- (ii) *When $\xi = \infty$, $\mathbf{I}$ and $\mathbf{I}'$ are good*
- (iii) *When $\rho = 1$, Λ is good.*

Part (iii) of Corollary 15 is a corollary of Lemma 12 and Lemma 14 together: observe $1 - o(1) = \sum_{\mathbf{z} \text{ good}} \mathbf{Pr}(\mathbf{I}' = \mathbf{z}) = (1 + o(1)) \sum_{\mathbf{z} \text{ good}} \mathbf{Pr}(\Lambda = \mathbf{z})$.

In conjunction with Lemma 12, Corollary 15 will allow us to say that events in Λ and $\mathbf{I}$ have roughly the same probability.

Events in $\mathbf{I}$, $\mathbf{I}'$, and Λ , are subsets of Ω_k , the set of all possible weightings on the $\binom{k}{2}$ edges. Thus, Ω_k is the set of all $\binom{k}{2}$ -vectors with non-negative

²Note, although the lemma is stated for a visit to a vertex v , v can also be a contraction of a subset of S .

entries. An event $\mathcal{E} \subseteq \Omega_k$ in $\mathbf{I}$ occurs iff $\mathbf{I} = \mathbf{z}$ where $\mathbf{z} \in \mathcal{E}$, indicated by $\mathbf{1}_{\mathcal{E}}(\mathbf{z})$. Similarly with $\mathbf{I}'$ and Λ .

Recall Λ depends on n . Lemma 16 below holds for the SI model; when we consider the case $\xi < 1$ we will give a generalisation of it.

LEMMA 16. *Assume the conditions in section 3.*

Suppose there is a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ and a constant p_c such that in Λ , $\Pr(\mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$. Then in $\mathbf{I}$, $\Pr(\mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$.

PROOF. Let $\mathbf{1}_{\mathcal{E}_n}(\mathbf{z})$ be the indicator that $\mathbf{z} \in \mathcal{E}_n$. By Lemma 12 and (ii) and (iii) of Corollary 15,

$$\begin{aligned} \Pr(\mathcal{E}_n) &= \sum_{\mathbf{z}} \Pr(\Lambda = \mathbf{z}) \mathbf{1}_{\mathcal{E}_n}(\mathbf{z}) \\ &= o(1) + \sum_{\mathbf{z} \text{ good}} \Pr(\Lambda = \mathbf{z}) \mathbf{1}_{\mathcal{E}_n}(\mathbf{z}) \\ &= o(1) + (1 - o(1)) \sum_{\mathbf{z} \text{ good}} \Pr(\mathbf{I}' = \mathbf{z}) \mathbf{1}_{\mathcal{E}_n}(\mathbf{z}) \\ &= o(1) + (1 - o(1)) \sum_{\mathbf{z}} \Pr(\mathbf{I}' = \mathbf{z}) \mathbf{1}_{\mathcal{E}_n}(\mathbf{z}). \end{aligned}$$

Hence in $\mathbf{I}'$, $\Pr(\mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$. Consequently, by part (i) of Corollary 15, $\Pr(\mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$ in $\mathbf{I}$. $\square$

9.3. *Completion time.* In [10] an expectation of $\frac{2\theta_r n}{k} \ln k$ was determined for the completion time of a broadcasting model on k particles that is equivalent to the SI model with $\rho = 1$. In section 11 the role of ψ will be made clear. Subsequently, the generalisation of the expectation to $\frac{2\theta_r n}{\psi k} \ln k$ for $\rho \leq 1$ will be seen to be straightforward. In this section, we shall get a convergence in probability to the same value, for the case $\rho = \psi = 1$. Subsequent to the treatment in section 11, it will be clear how the result extends to the general case.

We make use of a theorem from [19]: Assign each edge (i, j) of a complete graph on k vertices a random weight Y_{ij} . The weights are assumed to be independent and identically distributed, non-negative and satisfying $\Pr(Y_{ij} \leq t) = t + o(t)$ as $t \rightarrow 0$. Let X_{ij} be the minimal total weight of a path between a given pair of vertices i, j .

THEOREM 17 ([19]). *Under the assumptions above, for any fixed i , as $k \rightarrow \infty$,*

$$(32) \quad \frac{\max_j X_{ij}}{\ln k/k} \xrightarrow{p} 2.$$

We apply this below.

PROOF OF THEOREM 2 FOR $\rho = 1$. If a random variable $Y \sim \text{Exp}(\lambda)$, i.e., has exponential distribution with parameter λ , then $\lambda Y \sim \text{Exp}(1)$, which is valid for Theorem 17. Hence consider a complete graph F on the particle set with i.i.d. edge weights $Y_{ij} \sim \text{Exp}(\lambda)$, and let X_{ij} be the minimal total weight of a path between a given pair of vertices (particles) i, j in F . By the above, as $k \rightarrow \infty$,

$$(33) \quad \frac{\max_j \lambda X_{x_0j}}{\ln k/k} \xrightarrow{p} 2.$$

Let $\lambda = 1/(\theta_r n)$, so $w_\Lambda(i, j) \sim \text{Geom}(\lambda)$ for an edge $e = (i, j)$ in Λ . For each particle pair $e = (i, j)$, let U_{ij} be i.i.d random variables uniform on $[0, 1]$. We use U_{ij} to determine both Y_{ij} in F and $w_\Lambda(i, j)$ in Λ . Specifically, let $Y_{ij} = -\frac{\ln(U_{ij})}{\lambda}$ and let $w_\Lambda(i, j) = \left\lceil \frac{\ln(U_{ij})}{\ln(1-\lambda)} \right\rceil = \left\lceil \frac{-\ln(U_{ij})}{(1+O(1/n))\lambda} \right\rceil$. Hence $|Y_{ij} - w_\Lambda(i, j)| \leq 1$ for all pairs i, j when n is large enough, in which case, $|X_{x_0j} - d_\Lambda(j)| < k^2$. Thus,

$$\frac{\max_j \lambda(X_{x_0j} - k^2)}{\ln k/k} \leq \frac{\max_j \lambda d_\Lambda(j)}{\ln k/k} \leq \frac{\max_j \lambda(X_{x_0j} + k^2)}{\ln k/k}.$$

but $\lambda k^3 / \ln k = O(k^3/n)$. Hence, since $k \leq n^\epsilon$, when ϵ is small enough,

$$\frac{\max_j d_\Lambda(j)}{n \ln k/k} \xrightarrow{p} 2\theta_r.$$

By Lemmas 16 and 10, the lemma follows. $\square$

10. The SI(R) Model with $\rho = 1$. Now we allow $\xi < \infty$, thus generalising the previous section. We create $\mathbf{I}$ and $\mathbf{I}'$ exactly as before, and we will get particular realisations of each. As before, for a particular realisation $\mathbf{I} = \mathbf{z}$ (or $\mathbf{I}' = \mathbf{z}$), $\mathbf{z}$ is a $\binom{k}{2}$ -vector, however, recalling the definitions from section 8, we see that now some of the entries may be ∞ , whereas before in the SI case, they were all finite almost surely.

We define the function $f_\xi(F)$ that takes as input a weighted graph F and returns the same graph but with all edges with weights exceeding ξ deleted. We may also write $f_\xi(\mathbf{z})$, interpreting the argument as some graph weighted by $\mathbf{z}$.

For a graph $F = (\mathcal{P}, E_F)$, denote by $\mathcal{C}_F$ the connected component in F that x_0 belongs to. For example, take a complete graph on $\mathcal{P}$ weighted with $\mathbf{z}$ and apply $f_\xi(\mathbf{z})$ to delete edges with weight exceeding ξ . The connected component that x_0 belongs to in this graph is denoted by $\mathcal{C}_{f_\xi(\mathbf{z})}$.

The following lemma tells us which particles ever get infected.

LEMMA 18. *Let $F \in \{\mathbf{I}, \mathbf{I}'\}$. A particle y becomes infected iff $y \in \mathcal{C}_{f_\xi(F)}$.*

PROOF. If $y \in \mathcal{C}_{f_\xi(F)}$ then there is a path from x_0 to y with each edge having weight at most ξ . Each such edge can only exist because one of the ends was infected, and they subsequently interacted within ξ steps after the infection, meaning that the other will be infected after the interaction. Hence, y got infected.

Conversely, suppose y got infected. Then there was a chain of infections from x_0 to y . This would define a path with each edge weight at most ξ . Hence it would be in $\mathcal{C}_{f_\xi(F)}$. $\square$

By Lemma 18, we can determine the size of the outbreak via $\mathcal{C}_{f_\xi(\mathbf{I})}$ (of course, for the SI case, we did not need to consider $\mathcal{C}_{f_\xi(\mathbf{I})}$ since all edge weights would be finite and so none would get deleted). As before, we use the blind periods in order to apply the tools of section 7. Thus we work with $\mathbf{I}'$.

For a weighted graph $F = (\mathcal{P}, E_F)$ and a partition A, B of $\mathcal{P}$, denote by $E_F(A)$ and $E_F(B)$ the (weighted) edges induced by A and B respectively. Denote by $E(A : B)$ the (weighted) edges with one end in A and the other in B . The weights are explicitly part of the notation, thus, for example, if we write $E_{\mathbf{I}'}(A) = E_{\mathbf{z}}(A)$ for some subset A of the particles, we equate both the set of edges induced, as well as their weights.

In $\mathbf{I}'$, let A and B be the infected and non-infected particles, respectively. Observe that all weights in $E_{\mathbf{I}'}(B)$ are infinite, all in $E_{\mathbf{I}'}(A : B)$ are finite but larger than ξ , and all weights in $E_{\mathbf{I}'}(A)$ are finite, with some being at most ξ , and possibly some being greater than ξ .

The key observation is that *given edge weights $E_{\mathbf{I}'}(A)$ and $E_{\mathbf{I}'}(A : B)$, we know edge weights $E_{\mathbf{I}'}(B)$* . This is because we have timing information on the edges that allow us to reconstruct the unfolding of the process. Thus, determining the probability in $\mathbf{I}'$ of edge weights $E_{\mathbf{I}'}(A)$ and $E_{\mathbf{I}'}(A : B)$ is all that it required to determine the rest of the system, in particular, to show that those in A get infected and those in B do not.

Recall the definition of Λ from section 9.2. We reason as follows: Draw a Λ and delete all edges e with $w_\Lambda(e) > \xi$. The resulting component $\mathcal{C}_{f_\xi(\Lambda)}$ closely approximates $\mathcal{C}_{f_\xi(\mathbf{I})}$, informally justified as follows: Call a path an ξ -path if all edges in it have weight at most ξ . Partition $\mathcal{P}$ into A, B where $y \in A$ iff there is a ξ -path from x_0 to y in Λ . Observe in $E_\Lambda(A : B)$ all weights are greater than ξ , and in $E_\Lambda(A)$ there will be weights at most ξ , and possibly some exceeding ξ . All weights in these two sets will be finite. $E_{\mathbf{I}'}(A)$ and $E_{\mathbf{I}'}(A : B)$ will have roughly the same probability distribution on edge weights as $E_\Lambda(A)$ and $E_\Lambda(A : B)$, and since $\mathbf{I} = \mathbf{I}'$ **whp**, the same will

be true of $\mathbf{I}$. Therefore, $\mathcal{C}_{f_\xi(\mathbf{I})}$ has roughly the same probability distribution as $\mathcal{C}_{f_\xi(\Lambda)}$. Consequently, probabilistic statements about the size of $\mathcal{C}_{f_\xi(\Lambda)}$ can be transferred to $\mathcal{C}_{f_\xi(\mathbf{I})}$.

We proceed to formalise the above. We require the following generalisation of Lemma 11 (proof in the appendix). We denote by $V(\mathcal{C}_{f_\xi(\mathbf{z})})$ the set of vertices in $\mathcal{C}_{f_\xi(\mathbf{z})}$, i.e., $V(\mathcal{C}_{f_\xi(\mathbf{z})}) = \{x \in \mathcal{P} : \text{under } \mathbf{z} \text{ there is a } \xi\text{-path from } x_0 \text{ to } x\}$.

LEMMA 19. *Suppose $\mathbf{z}$ is good and let $A = V(\mathcal{C}_{f_\xi(\mathbf{z})})$, $B = \mathcal{P} \setminus A$. Consider the following:*

- (i) $\mathbf{I}'$ has $E_{\mathbf{I}'}(A) = E_{\mathbf{z}}(A)$ and $E_{\mathbf{I}'}(A : B) = E_{\mathbf{z}}(A : B)$
- (ii) For each particle pair (edge) $e = (x, y)$ such that $d_{f_\xi(\mathbf{z})}(e) < \infty$, there is no xy interaction in the period, $[d_{f_\xi(\mathbf{z})}(e) + \ell, d_{f_\xi(\mathbf{z})}(e) + w_{\mathbf{z}}(e) - 1]$ and there is an xy interaction at time $\tau(e) = d_{f_\xi(\mathbf{z})}(e) + w_{\mathbf{z}}(e)$

Then (i) holds if and only if (ii) holds.

Thus, we have established that (ii) implies the set $A = \mathcal{C}_{f_\xi(\mathbf{z})}$ gets infected and that for each edge e such that $d_{f_\xi(\mathbf{z})}(e) < \infty$, that is, for each edge $e \in E_{\mathbf{z}}(A) \cup E_{\mathbf{z}}(A : B)$, $w_{\mathbf{I}'}(e) = w_{\mathbf{z}}(e)$. It remains to establish that no particle in B is infected. If $d_{f_\xi(\mathbf{z})}(x) = \infty$, then x could not have been infected, since if it was then there must have been a chain of infections from x_0 to x . In this chain, there must have been some $x_b \notin \mathcal{C}_{f_\xi(\mathbf{z})}$ that was infected by some $x_a \in \mathcal{C}_{f_\xi(\mathbf{z})}$. But then $w_{\mathbf{z}}(x_a, x_b) > \xi$, and by (ii), this is when they first interact after x_a is infected. Hence, no infection could have been passed. Thus no particle in B is infected.

The following lemma generalises Lemma 12

LEMMA 20. *Assume the conditions of section 3.*

Suppose $\mathbf{z}$ is good. Define a partition of the particles into sets A and B where $A = V(\mathcal{C}_{f_\xi(\mathbf{z})})$. Let $\mathcal{S} = E_{\mathbf{z}}(A) \cup E_{\mathbf{z}}(A : B)$. Then

$$\Pr(E_{\mathbf{I}'}(A) \cup E_{\mathbf{I}'}(A : B) = \mathcal{S}) = (1 + o(1))\Pr(E_{\Lambda}(A) \cup E_{\Lambda}(A : B) = \mathcal{S}).$$

PROOF. The proof follows the same pattern as that of Lemma 12. Using Lemma 19, $E_{\mathbf{I}'}(A) \cup E_{\mathbf{I}'}(A : B) = \mathcal{S}$ defines a sequence of interactions $\sigma(\mathbf{z}) = \left((e_{(1)}, \tau_1), (e_{(2)}, \tau_2), \dots, (e_{\left(\binom{|S|}{2}\right)}, \tau_{\left(\binom{|S|}{2}\right)}) \right)$ and active edges sets $A_0, A_1, \dots, A_{|S|}$. We get an equivalent to (26),

$$(34) \quad \Pr(E_{\mathbf{I}'}(A) \cup E_{\mathbf{I}'}(A : B) = \mathcal{S}) = \prod_{j=0}^{\binom{|S|}{2}-1} p(1 - |A_j|p)^{\tau_{j+1} - \tau_j},$$

where

$$p = \frac{1}{\theta_r n} \left(1 + O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right).$$

We eventually get

$$\Pr(E_{\mathbf{I}'}(A) \cup E_{\mathbf{I}'}(A : B) = \mathcal{S}) = p^{\binom{|\mathcal{S}|}{2}} \exp \left\{ -(1 + O(k^2 p)) p \sum_{e \in \mathcal{S}} w_{\mathbf{z}}(e) \right\}$$

Again, $q = \frac{\psi}{\theta_r n}$ where here $\psi = 1$ because $\rho = 1$.

$$\begin{aligned} \Pr(E_{\Lambda}(A) \cup E_{\Lambda}(A : B) = \mathcal{S}) &= q^{|\mathcal{S}|} (1 - q)^{\sum_{e \in \mathcal{S}} w_{\mathbf{z}}(e) - |\mathcal{S}|} \\ &= q^{|\mathcal{S}|} (1 - q)^{-|\mathcal{S}|} \exp \left\{ -(1 + O(q)) q \sum_{e \in \mathcal{S}} w_{\mathbf{z}}(e) \right\}. \end{aligned}$$

Then taking the ratio and by the same reasoning as for Lemma 12, the result follows. $\square$

An event $\mathcal{E}$ for $\mathcal{C}_{f_{\xi}(\Lambda)}$ (or $\mathcal{C}_{f_{\xi}(\mathbf{I})}$) is a subset of the possible realisations of $\mathcal{C}_{f_{\xi}(\Lambda)}$ (resp. $\mathcal{C}_{f_{\xi}(\mathbf{I})}$). When we speak of a realisation, we include edge weights as well as graphical structure. For example, $S_g = \{\mathcal{C}_{f_{\xi}(\mathbf{z})} : \mathbf{z} \text{ good}\}$ is an event.

The following is a generalisation of Lemma 16 (proof in the appendix).

LEMMA 21. *Assume the conditions in section 3.*

Suppose there is a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ and a constant p_c such that $\Pr(\mathcal{C}_{f_{\xi}(\Lambda)} \in \mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$. Then $\Pr(\mathcal{C}_{f_{\xi}(\mathbf{I})} \in \mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$.

PROOF OF THEOREM 1 WHEN $\rho = 1$. Since $f_{\xi}(\Lambda)$ is an Erdős-Rényi random graph $\mathcal{G}_{k, \hat{q}}$ where $\hat{q} = 1 - (1 - \frac{1}{\theta_r n})^{\xi}$, we can apply standard results (see, e.g., [18], [20]) to determine the size of $\mathcal{C}_{f_{\xi}(\Lambda)}$. We address cases (i) and (iii) of Theorem 1 first. (i) If $k\hat{q} < 1$, then there is a constant α , such that $\Pr(|\mathcal{C}_{f_{\xi}(\Lambda)}| / \ln k \leq \alpha) \rightarrow 1$. Applying Lemma 21 for the events $\mathcal{E}_n = \{\mathcal{C}_{f_{\xi}(\mathbf{z})} : |\mathcal{C}_{f_{\xi}(\mathbf{z})}| / \ln k \leq \alpha\}$ gives the result. (iii) If $k\hat{q} > (1 + \varepsilon) \ln k$ where ε is any positive constant then $\Pr(|\mathcal{C}_{f_{\xi}(\Lambda)}| = k) \rightarrow 1$. Applying Lemma 21 for the events $\mathcal{E}_n = \{\mathcal{C}_{f_{\xi}(\mathbf{z})} : |\mathcal{C}_{f_{\xi}(\mathbf{z})}| = k\}$ gives the result.

Case (ii) requires slightly more consideration. We have $\mathcal{G}_{k, \hat{q}}$ where $k\hat{q} \rightarrow c$ for some constant $c > 1$. Denote the largest component by $\mathcal{C}_1$ and let $\beta = \beta(c)$ denote the unique solution in $(0, 1)$ of the equation $\beta + e^{-\beta c} = 1$. Then every $\nu \in (\frac{1}{2}, 1)$ there exists a $\delta = \delta(\nu, c)$ such that $\Pr\left(\left|\frac{|\mathcal{C}_1|}{k} - \beta\right| \leq k^{-\nu}\right) =$

$1 - O(k^\delta)$ (see, e.g., [18]). With high probability, all other components have size $O(\ln k)$. By the symmetry of Λ , $\Pr(\mathcal{C}_{f_\xi(\Lambda)} = \mathcal{C}_1) = \Pr(x_0 \in \mathcal{C}_1) = \mathbf{E}[|\mathcal{C}_1|]/k \rightarrow \beta$ as $k \rightarrow \infty$. Hence,

$$\begin{aligned} & \Pr\left(\left|\frac{|\mathcal{C}_{f_\xi(\Lambda)}|}{k} - \beta\right| \leq k^{-\nu}\right) \\ &= \Pr\left(\left|\frac{|\mathcal{C}_1|}{k} - \beta\right| \leq k^{-\nu} \wedge \mathcal{C}_{f_\xi(\Lambda)} = \mathcal{C}_1\right) + \Pr\left(\left|\frac{|\mathcal{C}_{f_\xi(\Lambda)}|}{k} - \beta\right| \leq k^{-\nu} \wedge \mathcal{C}_{f_\xi(\Lambda)} \neq \mathcal{C}_1\right) \\ &\rightarrow \beta. \end{aligned}$$

Applying Lemma 21 for the events $\mathcal{E}_n = \{\mathcal{C}_{f_\xi(\mathbf{z})} : |\mathcal{C}_{f_\xi(\mathbf{z})}|/k - \beta| \leq k^{-\nu}\}$, we get $\Pr\left(\left|\frac{|\mathcal{C}_{f_\xi(\Lambda)}|}{k} - \beta\right| \leq k^{-\nu}\right) \rightarrow \beta$, completing the proof of this case. $\square$

11. Extending to the General Case: SI(R) with $\rho \leq 1$. This section deals with general ξ and $\rho \leq 1$.

11.1. *A Heuristic Treatment of a Two-particle System.* We introduce the approach we shall use by giving an informal treatment for a system with only two particles. In subsequent sections, we will give a more rigorous analysis for k particles, where, as per the assumptions in section 3, $k \leq n^\epsilon$, for ϵ a sufficiently small constant.

Let x and y be the two particles, with x being the initial infective and y being susceptible. We allow $\xi < \infty$ and/or $\rho < 1$. The former conditions means that y may never get infected, the latter condition means that it may take more than one meeting between x and y before an interaction takes place. Note, that if x and y were at the same vertex at time t , and happen to move to the same neighbouring vertex in the next step, then this counts as another meeting, with another coin flip to determine if an interaction takes place.

Now, suppose x and y have just stepped to the same vertex v . With probability ρ there will be an interaction. After this, they will move again, either to the same neighbour of v with probability $1/r$ or to different neighbours with probability $(r-1)/r$. Let

$$\begin{aligned} \phi &= \Pr(\text{No } xy \text{ interaction before they move apart}) \\ &= \sum_{i \geq 1} (1-\rho)^i \left(\frac{1}{r}\right)^{i-1} \left(1 - \frac{1}{r}\right) = \frac{(1-\rho)(r-1)}{r-1+\rho}. \end{aligned}$$

Recall from section 7.1 that a vertex v is *treelike* if there is no cycle in the subgraph $G[v, L_1]$ induced by the set of vertices within (graph) distance

$L_1 = \lfloor \epsilon_1 \log_r n \rfloor$ of v , where $\epsilon_1 > 0$ is a sufficiently small constant. The following lemma is from [10]:

LEMMA 22 ([10]). *Let G be a typical r -regular graph, and let v be a vertex of G , treelike to depth $L_1 = \lfloor \epsilon_1 \log_r n \rfloor$. Suppose that at time zero, two independent random walks $(\mathcal{W}_1, \mathcal{W}_2)$ start from v . Let (a, b) denote the position of the particles at any step. Let $S = \{(u, u) : u \in V\}$. Let f be the probability of a first return to S within $T = O(k \ln n)$ steps given that the walks leave v by different edges at time zero. Then*

$$f = \frac{1}{(r-1)^2} + O\left(n^{-\Omega(1)}\right).$$

Using this lemma, let

$$\begin{aligned} \phi_T &= \Pr(\text{No } xy \text{ interaction before being apart more than } T \text{ time steps}) \\ (35) &= \sum_{i \geq 1} \phi^i f^{i-1} (1-f) = \frac{\phi(1-f)}{1-\phi f}. \end{aligned}$$

If two sequences A_n, B_n are such that $A_n/B_n \rightarrow 1$ as $n \rightarrow \infty$, we write $A_n \sim B_n$. The rest of this section will be implicitly justified in the detailed treatment in section 11.2.

Recall ψ was defined in (2), and observe that $\rho \leq \psi \leq 1$ with $\psi = 1$ iff $\rho = 1$. Now, assuming x and y start at the same vertex, then as will be seen from Lemma 27,

$$\begin{aligned} &\Pr(xy \text{ interaction occurs within } T \text{ time steps}) \\ &\sim \Pr(xy \text{ interaction occurs before } x \text{ and } y \text{ have been apart more than } T \text{ steps}) \\ (36) &= 1 - \phi_T = 1 - \frac{\phi(1-f)}{1-\phi f} \sim \frac{\rho(r-1)}{r-2+\rho} = \psi. \end{aligned}$$

DEFINITION 3 (**ℓ -distinct**). *A sequence $(t_1, t_2, \dots)$ is ℓ -distinct if $t_1 \geq \ell$ and $t_{i+1} - t_i \geq \ell$.*

Clearly, there can be at most t/ℓ ℓ -distinct meetings in $[0, t]$, and assuming $i \leq t/\ell$,

$$(37) \quad \Pr(\text{there are } i \text{ } \ell\text{-distinct meetings in } [0, t]) \sim \binom{t}{i} p^i (1-p)^{t-i},$$

where $p = (1 + o(1)) \frac{1}{\theta_r n}$ is from (21). Hence, it is seen to be approximately distributed as $\text{Binom}(t, \frac{1}{\theta_r n})$. The probability that there are no interactions

in any of the i intervals $[t_j, t_j + T]$ where t_j is the time of the j 'th ℓ -distinct meeting is $(1 - \psi)^i$. Thus

$$\begin{aligned} \Pr(\text{there are no interactions in the period } [0, t]) &\sim \sum_{i=0}^{t/\ell} \binom{t}{i} (p(1 - \psi))^i (1 - p)^{t-i} \\ &\sim (1 - \psi p)^t. \end{aligned}$$

Hence,

$$(38) \quad \Pr(y \text{ gets infected within time } \xi) \sim 1 - (1 - \psi p)^\xi.$$

When $\rho = \psi = 1$, (38) looks similar to the bracketed terms in (3). This is, of course, not a coincidence since the bracketed term in (3) is essentially the probability that an infection is passed between a pair of particles if one of them had been infected. Therefore, ϕ is effectively the expected number of other particles that are infected by a particular particle.

11.2. Allowing $\rho < 1$. In this section, we formalise some of the ideas of section 11.1, extended to k particles.

Let us first redefine ℓ to be $\ell = 2(T + T^3) + T$ where T is a maximal mixing time.

Consider a period $[0, t]$. Let $\tau = (t_1, \dots, t_i, t_{i+1})$ be a sequence such that (i) $\sum_{s=1}^{i+1} t_s = t$, (ii) $t_s \geq \ell$ for each $s \in \{1, 2, \dots, i\}$, (iii) $t_i + t_{i+1} \geq \ell + T$ and (iv) $0 \leq t_{i+1} \leq T$.

Let $t_0^* = 0$ and $t_s^* = t_{s-1}^* + t_s$ for $s = 1, 2, \dots, i$. We use τ to represent having i ℓ -distinct meetings in the period $[0, t]$ with a first interaction at step t (recall definition of T -distinct given in Definition 3). Specifically, let A be a set of active particle pairs and let $(x, y) \in A$. Denote by $\mathcal{C}_{(x,y)}(\tau)$ the following event: (1) at each time t_s^* a single particle pair $(a, b)_s \in A$ meets and does not interact in the period $[t_s^*, t_s^* + T]$. (2) At time t_i^* , (x, y) meet and interact at some point in the period $[t_i^*, t]$. (3) No particle pair $(a, b) \in A$ meets in the periods $[t_s^* + \ell, t_{s+1}^* - 1]$, $s = 0, 1, \dots, i$.

Note, in (2), we do not specify which particle pair (a, b) meet, only that a single pair meet. Thus, there are $|A|^{i-1}$ possible sequences of particles that satisfy this condition.

Let $\tau_i(t)$ be the set of all sequences $\tau = (t_1, \dots, t_i, t_{i+1})$ satisfying the above conditions, and let $\tau(t) = \cup_i \tau_i(t)$.

Define the event $\mathcal{C}_{(x,y)}(t) = \cup_{\tau \in \tau(t)} \mathcal{C}_{(x,y)}(\tau)$. Thus $\mathcal{C}_{(x,y)}(t)$ represents, subject to conditions (i)-(iv), having some number of non-interactive meetings of active particle pairs, and the first interaction taking place between x and y at time t .

The following Lemma is core to this section. The proof is rather long, so we delay it till later.

LEMMA 23. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Suppose $\sqrt{n} \leq t \leq k^2 n \ln n$. For a set of active particle pairs A and $(x, y) \in A$,

$$\Pr(\mathcal{C}_{(x,y)}(t)) = \psi p (1 - |A| \psi p)^t$$

where

$$(39) \quad p = \frac{1}{\theta_r n} \left(1 + O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right),$$

and

$$\psi = \frac{\rho(r-1)}{r-2+\rho}.$$

In the proofs of Lemma 12 and Lemma 20, we dealt with epochs τ_j and sets A_j of active edges. We assumed $\mathbf{z}$ was good and calculated the probability of a sequence $\sigma(\mathbf{z})$. This sequence specified that no particle pair in A_j would interact in the period $[\tau_j, \tau_{j+1} - 1]$ then a particular pair (x, y) interact at time τ_{j+1} . Because $\rho = 1$, interactions coincided with meetings, and so the sequence $\sigma(\mathbf{z})$ specified meetings and non-meetings. When $\rho < 1$, there may be some number of non-interactive meetings of active particle pairs in the period $[\tau_j, \tau_{j+1} - 1]$ before the xy interaction finally takes place at time τ_{j+1} , and we will need to take these into account when calculating the probability of $\sigma(\mathbf{z})$.

Recall that the raison d'être of *good* $\mathbf{z}$, rather than any arbitrary $\mathbf{z}$, was that having meetings take place ℓ steps apart (i.e., being ℓ -distinct) allowed us to apply Lemma 9. Although this lemma could tell us what happens in the period $[\tau_j + \ell, \tau_{j+1}]$, we could not account for what happens in the first ℓ steps after the epoch τ_j . We therefore modified the process in section 9.1 so that $\rho(t) = 0$ in these periods we could not account for, thereby guaranteeing that no interaction occurred and therefore, the probability calculated by Lemma 9 was faithful to the modified process. Hence we calculated probabilities for $\mathbf{I}'$ rather than $\mathbf{I}$, and we then related the two by proving they are the same **whp**. Since Lemma 9 is the main tool in the proof of Lemma 23, we must do something similar.

In this section, we need to have a slightly different version of the interaction graph $\mathbf{I}'$, which we shall denote by $\mathbf{I}^*$. To cope with sequences τ satisfying conditions (i)-(iv), we modify the process as follows (cf. section 9.1):

- (i) Set $\rho(t) = 0$ for $t \in [1, \ell]$
- (ii) If at time τ there was a meeting between one active pair $(x, y) \in A$, do the following: (a) For (x, y) , set $\rho(t) = 0$ for $t \in [\tau + T, \tau + \ell]$.
 (b) For every other $(a, b) \in A$, set $\rho(t) = 0$ for $t \in [\tau + 1, \tau + \ell]$.

Note, there remains the possibility of more than one active particle pair meeting at a particular time step. With high probability, this will not happen, as stated in Lemma 24. However, for completeness, we will include the following component to the above modification, which represents a ‘failure’:

- (iii) If at time τ there were meetings of more than one active pair in A , then set $\mathbf{I}^* = (-1, -1, \dots, -1)$ and terminate the construction.

Note, we did not need (iii) in the $\rho = 1$ analysis because we calculated meetings based on $\mathbf{z}$ being good, which required that no two pairs of active particles simultaneously meet, in accordance with part (a) of the definition of good (Definition 2). In this section, we consider meetings where no interaction takes places, and modification (iii) serves as a technical convenience.

The implication of these modifications is that in the application of Lemma 9 in the proof of Lemma 23, there were no interactions in the periods that we could not account for.

$\mathbf{I}^*$ is essentially a generalisation of $\mathbf{I}'$. For $\mathbf{I}'$, we did not need to stipulate (ii)(a) above, since if (x, y) met, they interacted, and further interaction between them thereafter had no bearing on edge weight $w_{\mathbf{I}'}(x, y)$. Hence, (ii)(a) would have been redundant.

The following lemma allows us to say that under these new modifications, **whp**, $\mathbf{I} = \mathbf{I}^*$. The proof is in the appendix.

LEMMA 24. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ . With high probability,*

- (a) *Only one pair meet at a time and no other pair meet within $\ell = 3T + 2T^3$ steps*
- (b) *Any active pair that meets at some step τ and does not interact in the period $[\tau, \tau + T - 1]$ does not meet in the period $[\tau + T, \tau + \ell]$*

COROLLARY 25. *With high probability, $\mathbf{I} = \mathbf{I}^*$.*

Consequently, if for a sequence of events $\mathcal{E}_k$ we determine that in $\mathbf{I}^*$, $\Pr(\mathcal{E}_k) \rightarrow p_c$ where p_c is a constant, then it will also be the case in $\mathbf{I}$ that $\Pr(\mathcal{E}_k) \rightarrow p_c$.

Observe an assumption in Lemma 23 is $\sqrt{n} \leq t \leq k^2 n \ln n$. The RHS inequality follows from the definition of good (section 9.2). The LHS inequality is an extra condition we must impose; thus we redefine Definition 2 part (a) as follows: *If $\mathbf{I} = \mathbf{z}$ then none of the $\binom{k}{2}$ interactions that form the edges of $\mathbf{I}$ took place within $\sqrt{n}$ steps of each other.* Showing that Lemma 14 still holds is straightforward: by Lemma 6 the probability of any of the at most $\binom{k}{2}$ particles meeting within $\sqrt{n}$ steps is $O(k^2 \sqrt{n}/n) = O(k^2/\sqrt{n})$. Taken over all (at most $\binom{k}{2}$) interactions this is $O(k^4/\sqrt{n}) = o(1)$ when $k \leq n^\epsilon$ for small enough ϵ .

We remind the definition of Λ given in section 9.2 stipulates i.i.d. edge weights distributed as $\text{Geom}(q)$ where $q = \psi/(\theta_\tau n)$.

LEMMA 26. *Assume the conditions of section 3.*

Suppose $\mathbf{z}$ is good. Define a partition of the particles into sets A and B where $A = V(\mathcal{C}_{f_\xi}(\mathbf{z}))$. Let $\mathcal{S} = E_{\mathbf{z}}(A) \cup E_{\mathbf{z}}(A : B)$. Then

$$\Pr(E_{\mathbf{I}^*}(A) \cup E_{\mathbf{I}^*}(A : B) = \mathcal{S}) = (1 + o(1)) \Pr(E_{\Lambda}(A) \cup E_{\Lambda}(A : B) = \mathcal{S}).$$

PROOF. Observe that Lemma 19 holds for $\mathbf{I}^*$ in the same way as it does for $\mathbf{I}'$. It is stated and proved in terms of interactions; non-interactive meetings are irrelevant. Hence, we can now follow the pattern of the proof of Lemma 20: $E_{\mathbf{I}^*}(A) \cup E_{\mathbf{I}^*}(A : B) = \mathcal{S}$ defines a sequence of interactions $\sigma(\mathbf{z}) = \left((e_{(1)}, \tau_1), (e_{(2)}, \tau_2), \dots, (e_{\binom{|S|}{2}}, \tau_{\binom{|S|}{2}}) \right)$ and active edges sets $A_0, A_1, \dots, A_{|S|}$.

Consider the period between τ_j and τ_{j+1} for $1 \leq j \leq \binom{|S|}{2} - 1$. By construction of $\mathbf{I}^*$, specifically, by the process modifications (i) and (ii) above, meetings in the period $[\tau_j, \tau_{j+1}]$ must be of the form τ satisfying conditions (i)-(iv) above, except those for which $t_i + t_{i+1} < \ell + T$, that is, those for which $t_{i-1}^* \in [t - (\ell + T), t - \ell]$. This window of size T can be ignored since, from the stationary distribution, the processes has probability $O(k^2 T/n)$ of falling into it. Taken over all (at most $\binom{k}{2}$) periods this is still $o(1)$ for $k \leq n^\epsilon$ and ϵ small enough.

We apply Lemma 23 to each period $[\tau_j, \tau_{j+1}]$ get to get an equivalent to (34):

$$(40) \quad \Pr(E_{\mathbf{I}^*}(A) \cup E_{\mathbf{I}^*}(A : B) = \mathcal{S}) = \prod_{j=0}^{\binom{|S|}{2}-1} \psi p (1 - |A_j| \psi p)^{\tau_{j+1} - \tau_j},$$

where

$$p = \frac{1}{\theta_r n} \left(1 + O \left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} \right) \right).$$

Recalling $q = \frac{\psi}{\theta_r n}$, the rest of the proof follows in the same way as in Lemma 20 (and in turn, Lemma 12). $\square$

Note, without modification (iii), which sets $\mathbf{I}^*$ to a vector of -1 's we would have to consider sequences which give rise to $\mathbf{I}^* = \mathbf{z}$ and in which more than one active particle meet. Thus modification (iii) is a technical convenience.

With the proof of Lemma 26, the proofs of Theorems 1 and 2 follow for the general case $\rho \leq 1$ as they did for the special case $\rho = 1$. We address Theorem 2 first.

PROOF OF THEOREM 2. Recall that Lemma 14 is general, holding for $\xi \leq \infty$ and $\rho \leq 1$. Thus, if $\xi = \infty$, $\mathbf{I}$ remains good, **whp**, when $\rho < 1$. By Corollary 25, the same is true for $\mathbf{I}^*$. Consequently, by Lemma 26, Λ is good **whp**. Thus, we have all three components of Corollary 15 holding for $\rho \leq 1$. As such, Lemma 16 holds for $\xi = \infty$ and $\rho \leq 1$. Therefore, the proof of Theorem 2 given in section 9.2 holds with $\lambda = \psi/(\theta_r n)$. $\square$

PROOF OF THEOREM 1. Observe that the proof of Lemma 21 used Corollary 15 and Lemma 20. We have generalised both of these in this section, so Lemma 21 holds for the general case $\xi \leq \infty$, $\rho \leq 1$ in the same way, with the edges of Λ being i.i.d. as $\text{Geom}(\psi/(\theta_r n))$. Consequently, the general case of Theorem 1 is justified in the same way as the special case was in section 10 (observe that now $f_\xi(\Lambda)$ is an Erdős-Rényi random graph $\mathcal{G}_{k,\hat{q}}$ where $\hat{q} = 1 - (1 - \frac{\psi}{\theta_r n})^\xi$). $\square$

Before we proceed to prove Lemma 23, we require the following:

LEMMA 27. *Let G be typical and let $k \leq n^\epsilon$ for sufficiently small ϵ .*

Suppose that at time zero, two particles x, y positioned on a treelike vertex v interact with probability ρ . Let ψ' be the probability of an xy interaction in $[0, T - 1]$ where T is a mixing time. Then

$$(41) \quad \psi' = \psi \left(1 - O(n^{-\Omega(1)}) \right)$$

where $\psi = \frac{\rho(r-1)}{r-2+\rho}$.

PROOF. x and y start at the same vertex at time $t = 0$, and interact with probability ρ at time 0. Suppose the first xy interaction occurs at time $\tau \geq 0$. Now suppose x and y are incident (i.e., at the same vertex) at times $0 = t_0, t_1, \dots$. Let t_M be the smallest t_r in this sequence such that $t_{M+1} - t_M \geq T$. Hence, $t_M + T - 1$ is the first time that x and y have been apart $T - 1$ steps.

We first demonstrate that $\Pr(\tau < T) = \Pr(\tau < t_M + T) + o(1)$. Observe

$$\Pr(\tau \geq t_M + T) \leq \Pr(\tau \geq T) \leq \Pr(\tau \geq t_M + T) + \Pr(\{x, y \text{ meet in } [T, 2T]\}).$$

Using (11),

$$\begin{aligned} \Pr(x, y \text{ meet in } [T, 2T]) &= 1 - \frac{(1 + O(T\pi_\gamma))}{(1 + (1 + O(T\pi_\gamma))\pi_\gamma/R_\gamma)^{2T}} - O\left(T^2\pi_\gamma e^{-\lambda T}\right) \\ &= O(T^2\pi_\gamma) \\ &= O((k \ln n)^2/n). \end{aligned}$$

Thus

$$\Pr(\tau \geq t_M + T) \leq \Pr(\tau \geq T) \leq \Pr(\tau \geq t_M + T) + O((k \ln n)^2/n),$$

i.e., $\Pr(\tau < t_M + T) = \Pr(\tau < T) + O((k \ln n)^2/n)$.

Recall section 11.1. To get the correcting factor in (35), observe that the sum $\sum_{i \geq 1} \phi^i f^{i-1} (1 - f)$ assumes every vertex at which the particles part is tree-like. Let $\mathcal{W}(t)$ be the walks on G and let $\mathcal{X}(t)$ be a walk on an infinite r -regular tree $\mathcal{T}$ rooted at the start vertex v , which is assumed to be tree-like. We couple $\mathcal{W}$ and $\mathcal{X}$ until time L_1 . Since G and $\mathcal{T}$ have the same structure out to L_1 , the two processes are identical until $t = L_1$. Let $Y_t = \text{dist}(x, y)$ in G . It is shown in [10], proof of Lemma 17, that $\Pr(Y_{L_1} \leq L_1/2) = O(n^{-\Omega^+(1)})$ where the $\Omega^+(1)$ is an arbitrarily large constant. It is also shown that, subject to $k \leq n^\epsilon$ for a sufficiently small ϵ , $\Pr(\text{the walks meet in } [L_1, T] \text{ and } Y_{L_1} > L_1/2) = O(n^{-\Omega(1)})$. Thus,

$$\phi_T = \frac{\phi(1-f)}{1-\phi f} + O(n^{-\Omega(1)}).$$

Now we include the error term for the asymptotic equality in line (36).

$$\begin{aligned} (42) \quad 1 - \phi_T &= \frac{\rho(r-1)}{r-2+\rho} \left(1 + O\left(n^{-\Omega(1)}\right)\right) \\ &= \psi \left(1 + O\left(n^{-\Omega(1)}\right)\right) = \Pr(\tau < t_M + T). \end{aligned}$$

Note in (42) we have used the assumption that ρ is a constant to absorb (functions of) it into the O term. Thus

$$\mathbf{Pr}(\tau < T) = \mathbf{Pr}(\tau < t_M + T) - O\left(\frac{(k \ln n)^2}{n}\right) = \psi \left(1 + O\left(n^{-\Omega(1)}\right)\right) - O\left(\frac{(k \ln n)^2}{n}\right).$$

ρ is a constant means ψ is a constant and so defining $\psi' = \mathbf{Pr}(\tau < T)$,

$$\psi' = \psi \left(1 - O\left(\frac{1}{n^{\Omega(1)}}\right)\right).$$

□

PROOF OF LEMMA 23. Writing

$$\mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) = \sum_{\tau \in \tau_i(t)} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau)),$$

we have

$$\begin{aligned} \mathbf{Pr}(\mathcal{C}_{(x,y)}(t)) &= \sum_{i \geq 1} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) \\ (43) \quad &= \sum_{i=1}^{(k \ln n)^5} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) + \sum_{i \geq (k \ln n)^5 + 1} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))). \end{aligned}$$

We shall focus on the first sum, returning to the second later.

We can write $\psi' = \sum_{i=0}^{T-1} \rho_i$ where ρ_i is the probability that the first interaction happens at step i . So, for example, $\rho_0 = \rho$. For a given τ , let $\rho\tau = \rho_{t_{i+1}}$.

We shall calculate $\mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau))$. By Lemma 9, the probability that no pair in A meet in the period $[t_{s-1}^* + \ell, t_s^* - 1]$, then some *particular* pair $(a, b) \in A$ meet (and no others do) at time t_s^* is given by

$$\mathbf{Pr}(\mathcal{B}_{(a,b)}(t_s)) = \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}} + \frac{k^4 t_s}{n^2}\right)\right) p(1 - |A|p)^{t_s},$$

where

$$p = \frac{1}{\theta_r n} \left(1 + O\left(\frac{1}{n^{\Omega(1)}} + \frac{1}{(k \ln n)^{\Omega^+(1)}}\right)\right).$$

Since $t_s \leq t \leq k^2 n \ln n$, we can write $\mathbf{Pr}(\mathcal{B}_{(a,b)}(t_s)) = p(1 - |A|p)^{t_s}$. Taking the product over all i ℓ -distinct meetings we get

$$(44) \quad \prod_{s=1}^i p(1 - |A|p)^{t_s}.$$

In the first $i - 1$ of these the pair do not interact in the following T steps so we multiply (44) by $(1 - \psi')^{i-1}$, and the interaction happens at time t , which is $t_{i+1} \leq T$ after t_1^* , so we multiply by $\rho_{t_{i+1}}$. Thus, we get

$$(45) \quad \rho_{\boldsymbol{\tau}}(1 - \psi')^{i-1} \prod_{s=1}^i p(1 - |A|p)^{t_s}.$$

In the above expression, a particular pair is specified at each ℓ -distinct meeting. We wish to fix only the final pair, giving a total of $|A|^{i-1}$ possible cases. Each case has the above probability, so in total, we get

$$\begin{aligned} \Pr(\mathcal{C}_{(x,y)}(\boldsymbol{\tau})) &= |A|^{i-1} \rho_{\boldsymbol{\tau}}(1 - \psi')^{i-1} \prod_{s=1}^i p(1 - |A|p)^{t_s} \\ &= p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t \rho_{\boldsymbol{\tau}}(1 - |A|p)^{-t_{i+1}} \end{aligned}$$

Since $(1 - |A|p)^{-t_{i+1}} = 1 + O(Tk^2/n)$, it can be absorbed into the correcting factor in p , so we have

$$\Pr(\mathcal{C}_{(x,y)}(\boldsymbol{\tau})) = \rho_{\boldsymbol{\tau}} p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t.$$

Observe that for a given t , fixing $t_1, \dots, t_{i-1}$ and allowing t_{i+1} to vary from 0 to T , determines t_i . Letting $t_i^{(r)} = t - (t_1 + \dots + t_{i-1} + r)$, we have

$$\begin{aligned} \sum_{r=0}^T \Pr(\boldsymbol{\tau} = (t_1, \dots, t_{i-1}, t_i^{(r)}, r)) &= p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t \sum_{r=0}^T \rho_r \\ &= \psi' p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t. \end{aligned}$$

Recall $\boldsymbol{\tau}_i(t)$ is the set of all sequences $\boldsymbol{\tau} = (t_1, \dots, t_i, t_{i+1})$ satisfying conditions (i)-(iv). Let $\boldsymbol{\tau}_i^*(t) \subset \boldsymbol{\tau}_i(t)$ be those for which $t_{i+1} = 0$.

$$\begin{aligned} \Pr(\mathcal{C}_{(x,y)}(\boldsymbol{\tau}_i(t))) &= \sum_{\boldsymbol{\tau} \in \boldsymbol{\tau}_i(t)} \Pr(\mathcal{C}_{(x,y)}(\boldsymbol{\tau})) \\ &= \sum_{\boldsymbol{\tau} \in \boldsymbol{\tau}_i^*(t)} \psi' p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t \\ (46) \quad &= |\boldsymbol{\tau}_i^*(t)| \psi' p((1 - \psi')|A|p)^{i-1} (1 - |A|p)^t. \end{aligned}$$

Let $\mathcal{S}_i$ be the set of sequences for which $t_i < \ell + T$, or in which there is some $t_s < \ell$ for $1 \leq s \leq i - 1$.

$$\begin{aligned}
|\tau_i^*(t)| &= \binom{t}{i-1} - |\mathcal{S}_i| \\
&= \binom{t}{i-1} - O\left(\binom{t}{i-2} i\ell\right) \\
&= \binom{t}{i-1} \left(1 - O\left(\frac{i^2\ell}{t-i}\right)\right) \\
&= \binom{t}{i-1} \left(1 - O\left(\frac{(k \ln n)^{13}}{t}\right)\right).
\end{aligned}$$

The above holds because we are only considering $i \leq (k \ln n)^5$, and by assumption, $t \geq \sqrt{n}$, so when $k \leq n^\epsilon$ for small enough ϵ , we have $i = o(t)$. On the same basis, we can further reduce the fraction to $1/n^{\Omega(1)}$, which can be absorbed into the correcting factor of p .

Thus, continuing from (46),

$$\begin{aligned}
&\sum_{i=1}^{(k \ln n)^5} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) \\
&= \psi' p \sum_{i=1}^{(k \ln n)^5} (1 - |A|p)^{i-1} \binom{t}{i-1} ((1 - \psi')|A|p)^{i-1} (1 - |A|p)^{t-i+1} \\
&= \psi' p \sum_{i=1}^{(k \ln n)^5} \binom{t}{i-1} ((1 - \psi')|A|p)^{i-1} (1 - |A|p)^{t-i+1},
\end{aligned}$$

since $(1 - |A|p)^{i-1} = 1 + O(1/n^{\Omega(1)})$ when $i \leq (k \ln n)^5$.

Now

$$\sum_{i=1}^t \binom{t}{i-1} ((1 - \psi')|A|p)^{i-1} (1 - |A|p)^{t-i+1} = (1 - \psi'|A|p)^t - ((1 - \psi')|A|p)^t.$$

Since $\sqrt{n} \leq t$, $((1 - \psi')|A|p)^t = O((k^2/n)^{\sqrt{n}}) = O(1/n^{\Omega(\sqrt{n})})$.

Hence, putting the above into (43), we have

$$\begin{aligned}
\mathbf{Pr}(\mathcal{C}_{(x,y)}(t)) &= \psi' p (1 - \psi'|A|p)^t - O\left(\frac{1}{n^{\Omega(\sqrt{n})}}\right) \\
&+ \sum_{i \geq (k \ln n)^5+1} \mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) - \psi' p \binom{t}{i-1} ((1 - \psi')|A|p)^{i-1} (1 - |A|p)^{t-i+1}.
\end{aligned}$$

One may think of the sum term to be the error generated by approximating $\mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t)))$ with the binomial expression. Since $\mathbf{Pr}(\mathcal{C}_{(x,y)}(\tau_i(t))) \leq (1-\rho)^i$ and $(1-\psi') \leq (1-\rho)$, the absolute value of the sum term is at most

$$(47) \quad (1-\rho)^{(k \ln n)^5} \left(1 + \sum_{i=(k \ln n)^5}^t \binom{t}{i-1} (|A|p)^{i-1} (1-|A|p)^{t-i+1} \right).$$

Furthermore, $O((1-\rho)^{(k \ln n)^5}) = O(n^{-\Omega(k^5(\ln n)^4)})$, and

$$\begin{aligned} \sum_{i=(k \ln n)^5}^t \binom{t}{i-1} (|A|p)^{i-1} (1-|A|p)^{t-i+1} &\leq \sum_{i=(k \ln n)^5}^t \left(\frac{et}{i} \right)^i (|A|p)^i \\ &\leq \sum_{i=(k \ln n)^5}^{\infty} \left(\frac{et|A|p}{(k \ln n)^5} \right)^i \\ &= O \left(\left(\frac{k^2 t}{(k \ln n)^5 n} \right)^{(k \ln n)^5} \right). \end{aligned}$$

Since $t \leq k^2 n \ln n$, we have $\frac{k^2 t}{(k \ln n)^5 n} \leq \frac{1}{k(\ln n)^4}$. Therefore, (47) is $O(n^{-\Omega(k^5(\ln n)^4)})$. Hence, when $k \leq n^\epsilon$ for small enough ϵ ,

$$\begin{aligned} \mathbf{Pr}(\mathcal{C}_{(x,y)}(t)) &= \psi' p (1 - \psi' |A|p)^t - O \left(\frac{1}{n^{\Omega(\sqrt{n})}} \right) + O \left(\frac{1}{n^{\Omega(k^5(\ln n)^4)}} \right) \\ &= \psi' p (1 - \psi' |A|p)^t + O \left(\frac{1}{n^{\Omega(k^5(\ln n)^4)}} \right) \\ &= \left(1 + O \left(\frac{(1 - \psi' |A|p)^{-t}}{n^{\Omega(k^5(\ln n)^4)}} \right) \right) \psi' p (1 - \psi' |A|p)^t. \end{aligned}$$

Since $t \leq k^2 n \ln n$, we have for some constant C ,

$$(1 - \psi' |A|p)^{-t} = O((1 - Ck^2/n)^{-k^2 n \ln n}) = O((1 - Ck^2/n)^{-\frac{n}{Ck^2} k^4 \ln n}) = n^{O(k^4)}.$$

Therefore, we have $\mathbf{Pr}(\mathcal{C}_{(x,y)}(t)) = \psi' p (1 - \psi' |A|p)^t$.

Consider the correcting factor of (41); this can be absorbed in the above correcting factors, as well as into p . We are finally left with

$$\mathbf{Pr}(\mathcal{C}_{(x,y)}(t)) = \psi p (1 - |A|\psi p)^t.$$

□

12. Concluding Remarks.

12.1. *Comments on proof strategy.* In section 10, one may wonder why we do not employ the more obvious strategy of simply setting the infectious period to be infinite (i.e., letting it run as an SI process) then deleting from $\mathbf{I}$ edges with weight exceeding the original infectious period ξ . We could then have continued where section 9 left off, proving Theorem 1 without the need for the intervening material of section 10. Unfortunately, this approach fails, as can be demonstrated by the following example: Suppose there are four particles a, b, c, d with a being the initial infective. Suppose the following (*particlepair, timestep*) meetings take place (the repetition of cd is intentional): $(ab, 9), (ad, 11), (bc, 18), (cd, 22), (cd, 27), (ac, 100), (bd, 100)$. The SI-based interaction graph would have the following edge weights: $w_{\mathbf{I}}(a, b) = 9, w_{\mathbf{I}}(a, c) = 100, w_{\mathbf{I}}(a, d) = 11, w_{\mathbf{I}}(b, c) = 9, w_{\mathbf{I}}(b, d) = 91, w_{\mathbf{I}}(c, d) = 11$. Therefore, if $\xi = 10$ and we had a rule to remove any edge with weight greater than this, it would leave d isolated, suggesting it does not get infected. However, it clearly gets infected by the chain $a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d$.

In contrast, under the current scheme, we get weights $w_{\mathbf{I}}(a, b) = 9, w_{\mathbf{I}}(a, c) = 100, w_{\mathbf{I}}(a, d) = 11, w_{\mathbf{I}}(b, c) = 9, w_{\mathbf{I}}(b, d) = 91, w_{\mathbf{I}}(c, d) = 4$. Deleting edges, we see that all particles are in the connected component of a , meaning they all get infected.

12.2. *Extensions.* One obvious extension is generalising Theorem 2 to include the case $\xi < \infty$. This would require the maximum weighted distance from x_0 to other vertices in $\mathcal{C}_{f_\xi(\Lambda)}$. For this purpose, it may be possible to exploit recent results such as [3].

Another obvious extension would be making the infectious period random, independently for each particle. Of course, we would not be able to use the current strategy of deleting edges that exceed a particular finite weight, and it would appear that the techniques in this paper do not readily extend to be able to cope with this setting. If one were to relax the model to allow infectious periods to be associated with particle pairs rather than particles themselves, this would correspond to i.i.d. random cut-off thresholds on edges. However, it would be difficult to justify an interpretation of this model. It would probably be more fruitful to aim to calculate rough bounds on M_k rather than precise values we currently get.

Finally, one may consider other graph models. In particular, random graphs of a prescribed degree sequence generalise random regular graphs, so would seem an obvious extension. Such a model was studied in [1], and it would seem that some of the results and techniques in that paper could find use in a multiple walks setting.

Another setting is d -dimensional grids as per [27] and [22]. Those papers

study the SI model, getting results on broadcast time, which is equivalent to our completion time T_k . Studying the size of the outbreak in the SIR model is a natural avenue for investigation. The techniques in this paper would not be amenable to that setting, due to the very different nature of these families of graphs. For a pertinent example, random regular graphs are mostly locally treelike with short cycles being far from each other. Grids on the other hand, have many short cycles. Thus local behaviours of walks in the mixing times, (as well as the mixing times themselves) will be quite different.

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13. Appendix.

PROOF OF LEMMA 19. (i) $\Rightarrow$ (ii) It is straightforward to show that for a particle y , the infection time $t(y)$ is the weighted distance in $f_\xi(\mathbf{I}')$ between x_0 and y , which will be ∞ if there is no path. Therefore, (i) implies that for an edge $e = (x, y)$, $t(e) = d_{f_\xi(\mathbf{z})}(e)$, and by the construction of $\mathbf{I}'$, (ii) follows.

(ii) $\Rightarrow$ (i) Order particles in $\mathcal{C}_{f_\xi(\mathbf{z})}$ by their weighted distance from x_0 in $f_\xi(\mathbf{z})$: $x_0 = x_{(0)}, x_{(1)}, \dots, x_{(r-1)}$ where $r = |\mathcal{C}_{f_\xi(\mathbf{z})}|$ and $i < j \Rightarrow d_{f_\xi(\mathbf{z})}(x_{(i)}) \leq d_{f_\xi(\mathbf{z})}(x_{(j)})$. We shall prove that $t(x_{(i)}) = d_{f_\xi(\mathbf{z})}(x_{(i)})$ for $0 \leq i \leq r - 1$.

Clearly this proposition holds for $x_{(0)}$. Suppose for all $i \leq N - 1$, $t(x_{(i)}) = d_{f_\xi(\mathbf{z})}(x_{(i)})$. If $N = r$, we are done. Otherwise $N < r$. Let $x_{(M)}$ be a neighbour

of $x_{(N)}$ on a shortest path from x_0 to $x_{(N)}$. Since $d_{f_\xi(\mathbf{z})}(x_{(M)}) < d_{f_\xi(\mathbf{z})}(x_{(N)})$, $M < N$, so by the induction hypothesis, $t(x_{(M)}) = d_{f_\xi(\mathbf{z})}(x_{(M)}) = d_{f_\xi(\mathbf{z})}(e)$ where $e = (x_{(M)}, x_{(N)})$. By **(ii)** this implies $t(x_{(N)}) \leq d_{f_\xi(\mathbf{z})}(e) + w_{\mathbf{z}}(e) = d_{f_\xi(\mathbf{z})}(x_{(N)})$.

Now consider the chain of infections starting at x_0 that led to $x_{(N)}$ being infected. Let $x_{(j)}$ be the first in the chain where $j > N - 1$ and suppose it got infected by $x_{(i)}$, $i \leq N - 1$. Then by **(ii)**, $t(x_{(j)}) = d_{f_\xi(\mathbf{z})}(x_{(i)}) + w_{\mathbf{z}}(x_{(i)}, x_{(j)}) \geq d_{f_\xi(\mathbf{z})}(x_{(j)}) \geq d_{f_\xi(\mathbf{z})}(x_{(N)})$, implying $t(x_{(N)}) \geq d_{f_\xi(\mathbf{z})}(x_{(N)})$. Hence $t(x_{(N)}) = d_{f_\xi(\mathbf{z})}(x_{(N)})$ for $0 \leq i \leq r - 1$. $\square$

PROOF OF LEMMA 21. Let $A(\mathbf{z}) = V(\mathcal{C}_{f_\xi(\mathbf{z})})$, $B(\mathbf{z}) = \mathcal{P} \setminus A(\mathbf{z})$ and $E(\mathbf{z}) = E(A(\mathbf{z})) \cup E(A(\mathbf{z}) : B(\mathbf{z}))$. Let $E_\Lambda(\mathbf{z}) = E_\Lambda(A(\mathbf{z})) \cup E_\Lambda(A(\mathbf{z}) : B(\mathbf{z}))$, and similarly for $\mathbf{I}'$. Since Λ is good **whp**, by Lemma 20,

$$1 - o(1) = \sum_{E(\mathbf{z}): \mathbf{z} \text{ good}} \Pr(E_\Lambda(\mathbf{z}) = E(\mathbf{z})) = (1 - o(1)) \sum_{E(\mathbf{z}): \mathbf{z} \text{ good}} \Pr(E_{\mathbf{I}'}(\mathbf{z}) = E(\mathbf{z})).$$

Observe $\mathcal{C}_{f_\xi(\mathbf{z})} = \mathcal{C}_{f_\xi(E(\mathbf{z}))}$. Let $\mathbf{1}_{\mathcal{E}_n}(\mathcal{C}_{f_\xi(E(\mathbf{z}))})$ be the indicator for $\mathcal{C}_{f_\xi(E(\mathbf{z}))} \in \mathcal{E}_n$.

$$\begin{aligned} \Pr(\mathcal{C}_{f_\xi(\Lambda)} \in \mathcal{E}_n) &= \sum_{E(\mathbf{z})} \Pr(E_\Lambda(\mathbf{z}) = E(\mathbf{z})) \mathbf{1}_{\mathcal{E}_n}(\mathcal{C}_{f_\xi(E(\mathbf{z}))}) \\ &= o(1) + \sum_{E(\mathbf{z}): \mathbf{z} \text{ good}} \Pr(E_\Lambda(\mathbf{z}) = E(\mathbf{z})) \mathbf{1}_{\mathcal{E}_n}(\mathcal{C}_{f_\xi(E(\mathbf{z}))}) \\ &= o(1) + (1 - o(1)) \sum_{E(\mathbf{z}): \mathbf{z} \text{ good}} \Pr(E_{\mathbf{I}'}(\mathbf{z}) = E(\mathbf{z})) \mathbf{1}_{\mathcal{E}_n}(\mathcal{C}_{f_\xi(E(\mathbf{z}))}) \\ &= o(1) + (1 - o(1)) \sum_{E(\mathbf{z})} \Pr(E_{\mathbf{I}'}(\mathbf{z}) = E(\mathbf{z})) \mathbf{1}_{\mathcal{E}_n}(\mathcal{C}_{f_\xi(E(\mathbf{z}))}) \\ &= o(1) + (1 - o(1)) \Pr(\mathcal{C}_{f_\xi(\mathbf{I}')} \in \mathcal{E}_n). \end{aligned}$$

Hence if $\Pr(\mathcal{C}_{f_\xi(\Lambda)} \in \mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$ then $\Pr(\mathcal{C}_{f_\xi(\mathbf{I}')} \in \mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$. Corollary 15(i) says $\mathbf{I} = \mathbf{I}'$ **whp**, and so $\Pr(\mathcal{C}_{f_\xi(\mathbf{I})} \in \mathcal{E}_n) \rightarrow p_c$ as $n \rightarrow \infty$. $\square$

PROOF OF LEMMA 24. **(a)** In the proof of Lemma 14 we derived probabilities for each of these two types of “failures”: **(i)** more than one pair of active particles being within distance $d = \alpha(\ln \ln n + \ln k)$ of each other when some pair meet after the (extended) mixing time, and **(ii)** any pair meeting within ℓ steps when they all start with distance at least d from each other. These had probabilities $O(1/n^{\Omega(1)})$ and $O(1/(k \ln n)^{\Omega^+(1)})$, respectively. Each type of failure was considered over at most $\binom{k}{2}$ particle pairs

over at most $\binom{k}{2}$ meetings (interactions), so taking the union bound over all of them, the above probabilities were still $o(1)$.

Now consider the $\rho < 1$ setting. After the extended mixing time we wait for a meeting of an active pair. Say the active pair that meet is (x, y) . We allow (x, y) to have T steps to interact, before letting the system mix again for $2(T + T^3)$ steps, after which we wait for another active pair meeting. Thus, if there are no failures as described above, these first meetings after the extended mixing times are ℓ -distinct. Let the random variable X count the total number of such meetings over the course of the process, over all active particle pairs. The probability of an interaction in the T -length window is at least ρ . Since ρ is constant, $\mathbf{E}[X] = O(\binom{k}{2}/\rho) = O(k^2)$. Hence, the expected number of failures over all particle pairs over all X meetings is bounded by $\mathbf{E}[X(1/n^{\Omega(1)} + 1/(k \ln n)^{\Omega^+(1)})] = o(1)$.

(b) We can assume $\tau = 0$. Let E_1 be the event that x and y interact in the period $[0, T-1]$, and let E_2 be the event that x and y meet in the period $[T, \ell]$. As per Lemma 27, $\Pr(E_1) = \psi'$. Then $\Pr(E_2|\overline{E_1}) \leq \Pr(E_2)/\Pr(\overline{E_1}) = \Pr(E_2)/(1 - \psi')$.

Thus, since ψ' is almost a constant, $\Pr(E_2|\overline{E_1})$ is of the same order as $\Pr(E_2)$. The rest of the proof is similar to part (a). $\square$

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